Thesis for the degree of Master of Science in Mathematics

GAGA

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Introduction

The goal of this thesis is to prove the GAGA theorem which first appeared in Serre's paper *Géométrie algébrique et géométrie analytique* in 1956. In broad strokes, GAGA says that on a projective variety there is no difference between coherent analytic and algebraic sheaves. On those varieties it is thus possible to use analytic methods to solve algebraic problems, and vice versa. In particular, when the underlying variety is nonsingular the full arsenal of Hodge theory on Kähler manifolds can be brought to bear in the resolution of algebraic questions.

There is no denying that while it is certainly nice to know that coherent algebraic and analytic sheaves are virtually identical on projective varieties, then this doesn't do us much good unless we know what those words mean. For the benefit of the reader we thus include a quick review of the fundamentals of sheaf theory and algebraic and analytic geometry needed for the statement and proof of GAGA in chapters 1 and 2. Beware that our review is a long way from being self contained and makes heavy use of references to various sources found in the bibliography.

Chapters 3 and 4 form the heart of this thesis and are fairly detailed and explicit. The former is a small detour devoted to the proof of a theorem of Cartan and Serre, necessary for the proof of GAGA, while the latter contains the statement and proof of GAGA itself. Finally we have collected some algebraic facts used in this thesis in an appendix.

It should be noted that we state and prove GAGA in the context of algebraic varieties and complex spaces. Shortly after the publication of Serre's paper Grothendieck extended Serre's result to the category of schemes. While proving GAGA for this more general case would be quite nice, time constraints force us to settle for varieties and spaces.

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And finally, to my family, Björgvin, and Arnar: Takk fyrir allt.

Chapter 1

Sheaves

> The major virtue of sheaf theory is information-theoretic in nature. Most problems could be phrased and perhaps solved without sheaf theory, but the notation would be enormously more complicated and difficult to comprehend.
>
> R.O. Wells, *Differential analysis on complex manifolds.*

After a few applications of sheaf theory in the resolution of analytic problems, one quickly becomes convinced of the truth of Wells' words, and filled with little desire to solve the same problems without the use of sheaves. Since their introduction by Leray in the 1940's and their following development by Cartan, Oka, Serre and Grothendieck, sheaves have become a standard tool of analysis. This thesis will make much use of sheaf theory, but unfortunately it cannot serve as an introduction to the subject. Such an introduction, which applies sheaf theory to problems in complex analysis, may be found in [GR65].

We thus assume the reader is familiar with the concept of a sheaf over a topological space, but in this chapter we nevertheless collect some of the definitions and properties of sheaves and their cohomology which we will need later on. As coherent sheaves might be unfamiliar to some readers we also give a short introduction to them and develop some of their properties.

Our sheaves will always be sheaves of rings or sheaves of modules. Many of the results stated in this chapter apply to sheaves of more general algebraic structures as well. The interested reader can find further details in [God58].

1.1 Sheaves and cohomology

(1.1.1) Construction. Let $\mathcal{F}$ be a sheaf on X. We denote by $\mathcal{F}^{[0]}$ the sheaf of all sections of $\mathcal{F}$, even those which are not continuous. In other words, the set $\mathcal{F}^{[0]}(U)$ is the set of all maps $f : U \to \mathcal{F}$ such that $f(x) \in \mathcal{F}_x$ for all $x \in U$. We see that $\mathcal{F}^{[0]}$ is *flabby,* i.e. that any section on a set U can be extended to all of X, and that there is a canonical injection $j : \mathcal{F} \hookrightarrow \mathcal{F}^{[0]}$.

We define inductively a sequence of sheaves

$$0 \to \mathcal{F} \overset{d^0}{\to} \mathcal{F}^{[0]} \overset{d^1}{\to} \mathcal{F}^{[1]} \to ...$$

where $\mathcal{F}^{[q]} := (\mathrm{Coker}\, d^{q-1})^{[0]}$, and $d^q := j_{\mathrm{Coker}\, d^{q-1}} \circ \pi_q$. Here

$$\pi^q : \mathcal{F}^{[q-1]} \to \mathrm{Coker}\, d^{q-1}$$

is the standard projection and

$$j_{\mathrm{Coker}\, d^{q-1}} : \mathrm{Coker}\, d^{q-1} \hookrightarrow (\mathrm{Coker}\, d^{q-1})^{[0]}$$

is the canonical injection introduced above. We sometimes refer to the morphisms d^q as *differentials.*This sequence is exact by construction, and is called the *simplical flabby resolution* of $\mathcal{F}$. It first appeared in [God58].

Definition 1.1.2. We define the *q-th cohomology group* of X with values in $\mathcal{F}$ as

$$H^q(X, \mathcal{F}) := \frac{\mathrm{Ker}\,(d^q : \mathcal{F}^{[q]} \to \mathcal{F}^{[q+1]})}{\mathrm{Im}\,(d^{[q-1]} : \mathcal{F}^{[q-1]} \to \mathcal{F}^{[q]})}$$

with the convention that $H^q(X, \mathcal{F}) = 0$ whenever $q < 0$. The morphisms d_q induce morphisms of the cohomology groups, which we denote again by d_q in an abuse of notation.

We can also define the cohomology groups $H^q(X, \mathcal{F})$ via any exact sequence $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to ...$ where the sheaves $\mathcal{F}^q$ are flabby. The fact that the resulting cohomology groups are well defined, i.e. do not depend on the sequence in question, implies that this definition is equivalent to the one given above. This is done in detail in Godement's book which we cited above.

A quick induction argument now yields:

Proposition 1.1.3. *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves over X and $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then φ induces morphisms φ^q such that the diagram*

$$\begin{array}{ccc} H^q(X, \mathcal{F}) & \overset{d^q}{\to} & H^{q+1}(X, \mathcal{F}) \\ \varphi^q \downarrow & & \varphi^{q+1} \downarrow \\ H^q(X, \mathcal{G}) & \overset{\delta^q}{\to} & H^{q+1}(X, \mathcal{G}) \end{array}$$

is commutative for every $q \geq 0$.

Proposition 1.1.4. *To any exact sequence of sheaves* $0 \to \mathcal{F} \to \mathcal{S} \to \mathcal{G} \to 0$ *we can associate an exact sequence of cohomology groups*

$$0 \to \mathcal{F}(X) \to \mathcal{S}(X) \to \mathcal{G}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{S}) \to \dots$$

Sketch of proof: One quickly verifies that the induced morphisms

$$\varphi^q : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{S}) \text{ and } \psi^q : H^q(X, \mathcal{S}) \to H^q(X, \mathcal{G})$$

satisfy $\operatorname{Ker} \psi^q = \operatorname{Im} \varphi^q$ and that φ^0 is injective. The only problem is constructing the connecting morphism $\delta^q : H^q(X, \mathcal{G}) \to H^{q+1}(X, \mathcal{F})$, which is done with an application of the snake lemma from homological algebra, and one verifies that the resulting sequence is exact. □

Combining propositions 1.1.3 and 1.1.4 we get:

Proposition 1.1.5. *Suppose we have a commutative diagram of sheaves on* X

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A} & \to & \mathcal{B} & \to & \mathcal{C} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{F} & \to & \mathcal{S} & \to & \mathcal{G} & \to & 0 \end{array}$$

where the horizontal arrows are exact sequences. Then we have a commutative diagram of cohomology groups

$$\begin{array}{ccccccccc} \dots & \to & H^k(X, \mathcal{A}) & \to & H^k(X, \mathcal{B}) & \to & H^k(X, \mathcal{C}) & \to & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \to & H^k(X, \mathcal{F}) & \to & H^k(X, \mathcal{S}) & \to & H^k(X, \mathcal{G}) & \to & \dots \end{array}$$

for all $k \geq 0$.

Remark — We trust that readers are familiar with the Čech cohomology of a sheaf $\mathcal{F}$. While it is not always the case that we can identify Čech cohomology with the usual one, we do have the following theorem for acyclic coverings:

Theorem 1.1.6. (Leray) *Let* $\mathcal{F}$ *be a sheaf on* X *and let* $\mathcal{U} = (U_\alpha)$ *be a covering of* X. *Set* $U_{\alpha_0, \dots, \alpha_t} := U_{\alpha_0} \cap \dots \cap U_{\alpha_t}$ *for all* $t \geq 0$. *If*

$$H^s(U_{\alpha_0, \dots, \alpha_t}, \mathcal{F}) = 0$$

for all $s \geq 1$ *and all indices* $\alpha_0, \dots, \alpha_t$, *then*

$$\check{H}^k(\mathcal{U}, \mathcal{F}) \simeq H^k(X, \mathcal{F})$$

for all k, *where* $\check{H}^k(\mathcal{U}, \mathcal{F})$ *is the* k*-th Čech cohomology group of* $\mathcal{F}$ *with respect to the covering* $\mathcal{U}$.

We will need the following proposition in chapters 3 and 4 for certain arguments which proceed by descending induction on the order of a cohomology group. A proof of the proposition can for example be found in [Dem07].

Proposition 1.1.7. *Let $\mathcal{F}$ be a sheaf on a paracompact space X. Then $H^k(X, \mathcal{F}) = 0$ for $k > topdim\, X$.*

Lemma 1.1.8. *Let X be a topological space and $(U_\alpha)_{\alpha \in I}$ an open covering of X, and let $\mathcal{F}_\alpha$ be a sheaf of rings (resp. modules) over U_α for any $\alpha \in I$. Let $\theta_{\alpha\beta} : \mathcal{F}_\beta(U_\alpha \cap U_\beta) \to \mathcal{F}_\alpha(U_\alpha \cap U_\beta)$ be isomorphisms such that $\theta_{\alpha\beta} \circ \theta_{\beta\gamma} = \theta_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Then there exists a sheaf $\mathcal{F}$ over X and for each α an isomorphism $\eta_\alpha : \mathcal{F}_{|U_\alpha} \to \mathcal{F}_\alpha$ such that $\theta_{\alpha\beta} = \eta_\alpha \circ \eta_\beta^{-1}$ on $U_\alpha \cap U_\beta$.*

In slightly less obscure language, the lemma tells us that in order to define a sheaf on a topological space we only need to define it on an open covering of the space. We will need this lemma in chapter 2 when we construct a complex space out of an algebraic variety. The proof may be found in [Ser55].

1.2 Coherent sheaves

One of the more useful concepts in differential geometry is the one of a vector bundle over a manifold. However, vector bundles have a flaw; given two vector bundles and a morphism between them, then the kernel, image or cokernel of that morphism is in general not a vector bundle, which is a problem if we want to construct exact sequences of vector bundles. To solve this problem we must find a more general concept which is stable under the taking of kernels, images and cokernels.

We note that if E is a vector bundle of rank r with transition morphisms in a sheaf $\mathcal{O}$, then we can associate to E the sheaf of its sections $\mathcal{O}(E)$, which is a sheaf of $\mathcal{O}$-modules. It is in turn locally isomorphic to $\mathcal{O}^r$, or to put it another way, there is an exact sequence of sheaves

$$\mathcal{O}^r \to \mathcal{O}(E) \to 0$$

This suggests one way to generalize the concept of a vector bundle; we let the rank r vary between points. This leads us to the following concept:

Definition 1.2.1. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on X. Then $\mathcal{F}$ is *locally finitely generated* if for every point $x \in X$ there exists a neighborhood U of x and sections $f_1, \ldots, f_p \in \mathcal{F}(U)$ such that for every $y \in U$ the stalk $\mathcal{F}_y$ is generated by the germs $f_{1,y}, \ldots, f_{p,y}$ as a $\mathcal{O}$-module.

We see that this condition is equivalent to the existence of an exact sequence

$$\mathcal{O}^p_{|U} \to \mathcal{F}_{|U} \to 0$$

in a neighborhood of every point, where $\mathcal{F}_{|U}$ is the restriction of $\mathcal{F}$ to U, that is the union of all stalks $\mathcal{F}_x$ where $x \in U$. The point here is that the natural number p can vary depending on the point x. Unfortunately this is not enough to guarantee stability under the taking of kernels, images and cokernels of morphism, but this is almost enough. The final step needed is the concept of a coherent sheaf. First we define:

Definition 1.2.2. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on X and let $U \subset X$ be open. Let $f_1, \ldots, f_p$ be sections of $\mathcal{F}(U)$, then the kernel of the homomorphism

$$\mathcal{O}^{\oplus p}_{|U} \to \mathcal{F}_{|U}, \quad (g_{1,x}, \ldots, g_{p,x}) \mapsto \sum_{1 \leq j \leq p} g_{j,x} f_{j,x}$$

is a subsheaf $\mathcal{R}(f_1, \ldots, f_p)$ of $\mathcal{O}^{\oplus p}_U$, called the *sheaf of relations* between $f_1, \ldots, f_p$.

Definition 1.2.3. A sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on X is *coherent* if it satisfies the following conditions:

a) $\mathcal{F}$ is locally finitely generated,
b) for any open $U \subset X$ and any $f_1, \ldots, f_p \in \mathcal{F}(U)$ the sheaf of relations $\mathcal{R}(f_1, \ldots, f_p)$ is locally finitely generated.

Note that the second condition says that the kernel of any morphism $\varphi : \mathcal{O}^{\oplus p} \to \mathcal{F}$ is locally finitely generated. Also note that coherence is a local property.

The following theorem is the key to many good properties of coherent sheaves. A proof may be found in [Ser55].

Theorem 1.2.4. *Let $0 \to \mathcal{F} \to \mathcal{S} \to \mathcal{G} \to 0$ be an exact sequence of sheaves of $\mathcal{O}$-modules. If two of the sheaves are coherent, then the third one is coherent as well.*

Proposition 1.2.5. *Any locally finitely generated subsheaf $\mathcal{G}$ of a coherent sheaf $\mathcal{F}$ is coherent.*

Proof: We only need to show that for any sections $g_1, \ldots, g_p$ of $\mathcal{G}$ the sheaf of relations $\mathcal{R}(g_1, \ldots, g_p)$ is locally finitely generated. But this is true as the sections g_j are also sections of $\mathcal{F}$, which is coherent. □

The next few statements are quick corollaries of these two properties, mostly obtained by setting up appropriate short exact sequences, so we leave their proofs to the reader:

Corollary 1.2.6. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent sheaves. Then Ker φ, Im φ and Coker φ are coherent.*

Corollary 1.2.7. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent subsheaves of a coherent sheaf $\mathcal{S}$, then both $\mathcal{F} \cap \mathcal{G}$ and $\mathcal{S}/\mathcal{G}$ are coherent.*

Definition 1.2.8. A sheaf of rings $\mathcal{O}$ is said to be *coherent* if it is coherent as a module over itself.

Corollary 1.2.9. *If $\mathcal{O}$ is a coherent sheaf of rings, then $\mathcal{O}^{\oplus p}$ is coherent for all $p \geq 1$. Also, if $\mathcal{F}$ is a coherent $\mathcal{O}$-module and $f_1, \ldots, f_p \in \mathcal{F}(U)$, then the sheaf of relations $\mathcal{R}(f_1, \ldots, f_p)$ is coherent.*

Theorem 1.2.10. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules, where $\mathcal{O}$ is a coherent ring of sheaves. Then $\mathcal{F}$ is coherent if and only if for every integer $m \geq 0$ and every $x \in X$ there is a neighborhood U of x on which there is an exact sequence of sheaves*

$$\mathcal{O}_{|U}^{\oplus p_m} \to \mathcal{O}_{|U}^{\oplus p_{m-1}} \to \ldots \to \mathcal{O}_{|U}^{\oplus p_1} \to \mathcal{O}_{|U}^{\oplus p_0} \to \mathcal{F}_{|U} \to 0$$

Proof: Suppose that such a sequence exists in a neighborhood of every point and let x be a point of X. Then there certainly exists a neighborhood U of X and an exact sequence

$$\mathcal{O}_{|U}^{\oplus p_1} \to \mathcal{O}_{|U}^{\oplus p_0} \to \mathcal{F}_{|U} \to 0.$$

But as $\mathcal{O}^{\oplus p}$ is coherent for any $p \geq 1$, then $\mathcal{F}$ is locally isomorphic to the cokernel of a morphism of coherent sheaves, so $\mathcal{F}$ is coherent.

Now suppose that $\mathcal{F}$ is coherent. Then for any x we can find a neighborhood U_0 and a surjective morphism $\varphi_0 : \mathcal{O}_{|U_0}^{\oplus p_0} \to \mathcal{F}_{|U_0}$. Using the coherence of Ker φ_0, we can find a neighborhood $U_1 \subset U_0$ of x and a morphism φ_1 such that the sequence

$$\mathcal{O}_{|U_1}^{\oplus p_1} \xrightarrow{\varphi_1} \mathcal{O}_{|U_1}^{\oplus p_0} \xrightarrow{\varphi_0} \mathcal{F}_{|U_1} \to 0$$

is exact. We repeat this construction as many times as needed, and arrive at a neighborhood U of x on which there is an exact sequence as described in the statement of the theorem. $\square$

Note that the above theorem says in particular that a sheaf over a coherent sheaf of rings is coherent if and only if it is locally the cokernel of a morphism $\varphi : \mathcal{O}^{\oplus q} \to \mathcal{O}^{\oplus p}$. We will use this fact often.

Corollary 1.2.11. *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves of $\mathcal{O}$-modules. Then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is coherent.*

Proof: Note that in a neighborhood of every x in X we have an exact sequence

$$\mathcal{O}^{\oplus q} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0,$$

and thus an exact sequence

$$\mathcal{O}^{\oplus q} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{O}^{\oplus p} \otimes_{\mathcal{O}} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}} \mathcal{G} \to 0.$$

Now, $\mathcal{O}^{\oplus p} \otimes_{\mathcal{O}} \mathcal{G} = \mathcal{G}^{\oplus p}$ which is coherent, so $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is locally the cokernel of a morphism of coherent sheaves. □

Proposition 1.2.12. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules. If $\mathcal{F}$ is coherent, then we have an isomorphism*

$$Hom_{\mathcal{O}}(\mathcal{F}, \mathcal{G})_x \simeq Hom_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$$

Proof: There is a canonical morphism $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$: Let φ be an element of $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})_x$. On every small neighborhood U of x, the germ φ is given by a morphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, and with this collection of morphisms we can define a morphism $\varphi_x \in \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$.

Now take $\varphi : \mathcal{F} \to \mathcal{G}$ such that $\varphi_x = 0$. We want to show that $\varphi = 0$ on a neighborhood of x, or in other words, that the canonical morphism defined above is injective. As $\mathcal{F}$ is locally finitely generated, there exists a neighborhood U of x such that $\mathcal{F}$ is generated by sections $f_1, \dots, f_p$ on U. As $\varphi_x(f_{j,x}) = 0$ for all j, then $\varphi(f_j) = 0$ for all j on a neighborhood $V \subset U$ of x, but then $\varphi = 0$ on V.

Finally, let $\varphi_x \in \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$ be given. We want to find a morphism $\psi : \mathcal{F} \to \mathcal{G}$ such that $\psi_x = \varphi_x$. Now, let $f_1, \dots, f_p$ be sections of $\mathcal{F}$ that generate $\mathcal{F}$ near x, and let $F_j = (F_j^1, \dots, F_j^p)$, $1 \leq j \leq q$ be sections of $\mathcal{O}^{\oplus p}$ which generate $\mathcal{R}(f_1, \dots, f_p)$ near x. These sections exist because $\mathcal{F}$ is coherent. Also, let g_j be sections of $\mathcal{G}$ near x such that $g_{i,x} = \varphi_x(f_{i,x})$ for $1 \leq i \leq p$. We can write any section of $\mathcal{F}$ near x as $f = \sum a_i f_i$ where $a_i \in \mathcal{O}$, and we define our candidate for a map by $\psi(f) := \sum a_i g_i$. Obviously we have that $\psi_x = \varphi_x$, if our map ψ is well defined.

We thus need to show that if two different sums $\sum a_i f_i$ and $\sum a'_i f_i$ define the same germ f_x, then they map to the same element of $\mathcal{G}$, or, that if $a = (a_1, \ldots, a_p)$ is a section of $\mathcal{R}(f_1, \ldots, f_p)$ near x, then $\sum a_i f_i$ maps to zero. As

$$0 = \varphi_x \left(\sum_{i=1}^{p} F^i_{j,x} f_{i,x} \right) = \sum_{i=1}^{p} F^i_{j,x} \varphi_x(g_{i,x}) = \sum_{i=1}^{p} F^i_{j,x} g_{i,x}$$

and the right hand side is a germ of $\mathcal{G}$ at x, we conclude that ψ sends the generators of $\mathcal{R}(f_1, \ldots, f_p)$ to zero on a neighborhood of x. Therefore, ψ annihilates any linear combination of the generators, and thus all of the sheaf of relations. □

Corollary 1.2.13. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves of $\mathcal{O}$-modules, then $Hom(\mathcal{F}, \mathcal{G})$ is coherent.*

Proof: In a neighborhood of each x we have an exact sequence $\mathcal{O}^{\oplus q} \to \mathcal{O}^{\oplus p} \to \mathcal{F} \to 0$. By using the above proposition we obtain an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{O}^{\oplus q}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{O}^{\oplus p}, \mathcal{G}). \tag{1.1}$$

Now, for any p we have $\mathrm{Hom}(\mathcal{O}^{\oplus p}, \mathcal{G}) \simeq \mathcal{G}^{\oplus p}$, which is a coherent sheaf. The sheaf $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ is then equal to the kernel of a morphism of coherent sheaves. □

The following property of locally finitely generated sheaves is often useful:

Proposition 1.2.14. *Let $\mathcal{F}$ be a locally finitely generated sheaf. If $f_1, \ldots, f_p$ are sections of $\mathcal{F}$ that generate $\mathcal{F}_x$ at a point x, then they generate $\mathcal{F}_y$ for all y in a neighborhood of x.*

Proof: There is a neighborhood W of x and sections $g_1, \ldots, g_q$ of $\mathcal{F}(W)$ which generate $\mathcal{F}_y$ for $y \in W$. Since $f_{1,x}, \ldots, f_{p,x}$ generate $\mathcal{F}_x$ we can find sections h_{ij} in a neighborhood of x such that

$$g_{j,x} = \sum_{i=1}^{p} h_{ij,x} f_{i,x},$$

for $1 \leq j \leq q$. Therefore there exists a neighborhood U of x such that $g_j|U = \sum h_{ij} f_i$, and since $g_j|U$ generate $\mathcal{F}$ on U then f_i also generate $\mathcal{F}$ on U. □

Corollary 1.2.15. *If $\mathcal{F}$ is a locally finitely generated sheaf, then its support $Supp\,\mathcal{F} := \{x \in X | \, \mathcal{F}_x \neq 0\}$ is closed in X.*

Proof: The zero section of $\mathcal{F}$ generates the zero stalk, so the complement of the support is open. $\square$

This corollary along with the properties above of morphisms of coherent sheaves allow us to deduce the following result for coherent sheaves, which does not hold for arbitrary sheaves.

Corollary 1.2.16. *Let $\mathcal{F} \overset{\varphi}{\to} \mathcal{S} \overset{\psi}{\to} \mathcal{G}$ be a sequence of coherent sheaves which is exact at a point x. Then there exists a neighborhood U of x such that the sequence*

$$\mathcal{F}(U) \overset{\varphi_U}{\to} \mathcal{S}(U) \overset{\psi_U}{\to} \mathcal{G}(U)$$

is exact.

Proof: The sheaf $\mathcal{F}/\operatorname{Ker}\psi \circ \varphi$ is coherent and zero at x, so it is zero in an open neighborhood V of x, i.e. $\operatorname{Im}\varphi_V \subset \operatorname{Ker}\psi_V$. Similarly, the sheaf $\operatorname{Ker}\varphi_V/\operatorname{Im}\varphi_V$ is coherent and zero at x. Thus it is zero on a neighborhood $U \subset V$ of x, on which we then have $\operatorname{Ker}\psi_U = \operatorname{Im}\varphi_U$. $\square$

1.3 Extending and restricting a coherent sheaf

In chapter 4 we will be given a coherent sheaf $\mathcal{F}$ of $\mathcal{O}/\mathcal{I}$-modules over a closed subspace $Y \subset X$, where $\mathcal{O}$ is a coherent sheaf of rings and $\mathcal{I}$ is a coherent sheaf of ideals of $\mathcal{O}$. We will extend this sheaf to the whole space X and consider it as a sheaf of $\mathcal{O}$-modules. To this end we need to know to what extent these operations preserve coherence and cohomology.

A proof of the following proposition may be found in [God58] or [Dem07].

Proposition 1.3.1. *Let Y be a closed subspace of X. If $\mathcal{F}$ is a sheaf on Y, we denote by $\mathcal{F}^X$ its extension by 0 to the whole of X. We have an isomorphism of cohomology groups*

$$H^k(Y, \mathcal{F}) \simeq H^k(X, \mathcal{F}^X), \quad k \geq 0.$$

Proposition 1.3.2. *Let Y be a closed subspace of X. If $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules on Y, then $\mathcal{F}^X$ is a sheaf of $\mathcal{O}^X$-modules. The sheaf $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}$-modules if and only if $\mathcal{F}^X$ is a coherent sheaf of $\mathcal{O}^X$-modules.*

Proof: The sheaf $\mathcal{F}^X$ is clearly a sheaf of $\mathcal{O}^X$-modules. Let $U \subset X$ be open and set $V = Y \cap U$. If $\varphi : \mathcal{O}_{|V}^{\oplus p} \to \mathcal{F}_{|V}$ is a homomorphism, then we can define a homomorphism $\varphi^X : (\mathcal{O}^X)_{|U}^{\oplus p} \to \mathcal{F}_{|U}^X$ by setting $\varphi^X = 0$ outside of V. We also note that any homomorphism $\varphi^X : (\mathcal{O}^X)_{|U}^{\oplus p} \to \mathcal{F}_{|U}^X$ will be zero outside of V, because the sheaves in question are zero there, and thus we get a well defined morphism $\varphi : \mathcal{O}_{|V}^{\oplus p} \to \mathcal{F}_{|V}$. It is now clear that φ is surjective if and only if φ^X is surjective, which gives us that $\mathcal{F}$ is locally finitely generated if and only if $\mathcal{F}^X$ is locally finitely generated.

It rests to show that $\mathcal{R}(f_1, \ldots, f_q)$, where $f_j \in \mathcal{F}(V)$, is locally finitely generated if and only if $\mathcal{R}(f_1^X, \ldots, f_q^X)$, where $f_j^X \in \mathcal{F}^X(U)$, is locally finitely generated. The argument is essentially the same as the one above, so we leave it to the reader. □

Proposition 1.3.3. *Let $\mathcal{O}$ be a coherent sheaf of rings over X and let $\mathcal{I}$ be a coherent sheaf of ideals of $\mathcal{O}$. A sheaf $\mathcal{F}$ of $\mathcal{O}/\mathcal{I}$-modules is $\mathcal{O}/\mathcal{I}$-coherent if and only if it is $\mathcal{O}$-coherent.*

Proof: The map $\mathcal{O} \to \mathcal{O}/\mathcal{I}$ is surjective, so $\mathcal{F}$ is locally finitely generated as a $\mathcal{O}$-sheaf if and only if it is locally finitely generated as a $\mathcal{O}/\mathcal{I}$-sheaf.

Now let $f_1, \ldots, f_p$ be sections of $\mathcal{F}$ as a $\mathcal{O}/\mathcal{I}$-module over an open $U \subset X$, and let $\pi : \mathcal{O}^{\oplus p} \to (\mathcal{O}/\mathcal{I})^{\oplus p}$ be the projection. Then

$$\mathcal{R}(f_1, \ldots, f_p) = \pi(\tilde{\mathcal{R}}(f_1, \ldots, f_p))$$

where the sheaf of relations $\tilde{\mathcal{R}}$ is the subsheaf of $\mathcal{O}^{\oplus p}$ obtained by considering $f_1, \ldots, f_p$ as sections of $\mathcal{F}$ as a $\mathcal{O}$-module. If $\mathcal{F}$ is $\mathcal{O}$-coherent, then the $\mathcal{O}$-sheaf $\tilde{\mathcal{R}}(f_1, \ldots, f_p)$ is locally finitely generated, so the sheaf of relations $\mathcal{R}(f_1, \ldots, f_p)$ is locally finitely generated, and $\mathcal{F}$ is $\mathcal{O}/\mathcal{I}$-coherent.

Suppose that $\mathcal{F}$ is $\mathcal{O}/\mathcal{I}$-coherent. Then it is locally the cokernel of a map $\varphi : (\mathcal{O}/\mathcal{I})^{\oplus q} \to (\mathcal{O}/\mathcal{I})^{\oplus p}$. But $\mathcal{O}/\mathcal{I}$ is $\mathcal{O}$-coherent, so $\operatorname{Coker} \varphi \simeq \mathcal{F}$ is $\mathcal{O}$-coherent as well. □

Remark — Let $\mathcal{O}$ be a sheaf of rings and $\mathcal{I}$ an ideal sheaf of $\mathcal{O}$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}/\mathcal{I}$-modules, and denote by $\mathcal{F}_0$ the sheaf we obtain by considering $\mathcal{F}$ as a sheaf of $\mathcal{O}$-modules. Now note that $\mathcal{F}$ and $\mathcal{F}_{\mathcal{O}}$ have the same structure as $\mathbb{Z}$-modules. When we calculate the cohomology groups of a sheaf of modules we only use its $\mathbb{Z}$-module structure, so in fact we have that $H^k(X, \mathcal{F}) = H^k(X, \mathcal{F}_{\mathcal{O}})$ for all $k \geq 0$.

Definition 1.3.4. Let X and Y be topological spaces and $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a sheaf of rings over X, let $\mathcal{O}_Y$ be a sheaf of rings over Y, and suppose that $\mathcal{O}_X$ has the structure of a $f^{-1}\mathcal{O}_Y$-module. For a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$ modules over Y we define

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

and note that $f^*\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules over X.

Recall that there is a canonical morphism $\mathcal{F} \to f^{-1}\mathcal{F}$, so we obtain a canonical morphism $\mathcal{F} \to f^*\mathcal{F}$ by tensoring with the identity of $\mathcal{O}_X$.

The next couple of propositions will develop some properties of this modified inverse image operation. Unless we state otherwise, our notation will be the same as in definition 1.3.4.

Proposition 1.3.5. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then φ induces a morphism $f^*\varphi : f^*\mathcal{F} \to f^*\mathcal{G}$.*

Proof: We know φ induces a morphism $\tilde{\varphi} : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$. Set $f^*\varphi := \tilde{\varphi} \otimes_{f^{-1}\mathcal{O}_Y} \mathrm{id}_{\mathcal{O}_X}$. □

From the definition of the morphism $f^*\varphi$ we see that:

Corollary 1.3.6. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be morphisms of sheaves. Then $f^*(\psi \circ \varphi) = f^*\psi \circ f^*\varphi$.*

Note that the association $\mathcal{F} \mapsto f^{-1}\mathcal{F}$ preserves stalks and therefore exact sequences. Also note that the tensor product is right exact, i.e. if $\mathcal{F} \to \mathcal{S} \to G \to 0$ is exact, then $\mathcal{F} \otimes \mathcal{A} \to \mathcal{S} \otimes \mathcal{A} \to G \otimes \mathcal{A} \to 0$ is exact for any sheaf of modules $\mathcal{A}$. It follows that the operation $\mathcal{F} \mapsto f^*\mathcal{F}$ is right exact, and since $f^*(\mathcal{O}_Y^p) = \mathcal{O}_X^p$ we get:

Proposition 1.3.7. *If $\mathcal{F}$ is $\mathcal{O}_Y$-coherent, then $f^*\mathcal{F}$ is $\mathcal{O}_X$-coherent.*

Remark — We will use this operation in particular when X is a closed subspace of Y and f is the injection $\iota : X \hookrightarrow Y$. If $\mathcal{F}$ is a sheaf over Y, then $\iota^*\mathcal{F}$ will be the restriction of $\mathcal{F}$ to X but with the structure of a sheaf of $\mathcal{O}_X$-modules.

Chapter 2

Complex spaces and algebraic varieties.

GAGA is a theorem about coherent sheaves on algebraic and complex spaces, so we need to have some idea about what those things are. In vague but suggestive terms an algebraic variety is a topological space locally homeomorphic to the zero set of a finite number of polynomials, the homeomorphisms being given by so called regular maps, which are a slight generalization of rational functions. The idea of a complex space is a similar one, but there our space locally looks like the zero set of a finite number of holomorphic functions, and the homeomorphisms are holomorphic functions.

In this chapter we will make these notions more precise, but our treatment will be very incomplete and will only touch upon those definitions and properties needed to state and prove GAGA. For a more detailed discussion on algebraic varieties we refer to [Ser55] or [Per08], and to [Dem07] or [GR65] for background on complex spaces.

2.1 Complex spaces

Definition 2.1.1. A closed subset A of an analytic manifold M is an *analytic set* if for every x in A there exists a neighborhood U of x and holomorphic functions $f_1, \ldots, f_N$ on U such that

$$A \cap U = \{z \in U | \, f_1(z) = \ldots = f_N(z) = 0\}$$

Any closed submanifold of M is an analytic set, but an analytic set is not necessarily a submanifold of M because we do not require the differentials

$\mathrm{d}f_1, \ldots, \mathrm{d}f_N$ to be linearly independent. We nevertheless have a notion of a holomorphic function on an analytic set A:

Definition 2.1.2. For any $x \in M$ we let $\mathcal{I}_{A,x}$ be the ideal of germs $f \in \mathcal{O}_{M,x}$ such that f vanishes on A. We define $\mathcal{I}_A$ as the disjoint union of all $\mathcal{I}_{A,x}$. Then $\mathcal{I}_A$ is a subsheaf of $\mathcal{O}_M$, called the *ideal sheaf* of A. The *sheaf of holomorphic functions* on A is

$$\mathcal{O}_A := (\mathcal{O}_M/\mathcal{I}_A)|_A$$

The following two important theorems are due to Oka and Cartan, respectively, and combined with the results of chapter 1 they give us a great deal of information about the sheaves of holomorphic functions:

Theorem 2.1.3. (Oka) *The sheaf of rings $\mathcal{O}_M$ is coherent for any analytic manifold M.*

Theorem 2.1.4. (Cartan) *The ideal sheaf $\mathcal{I}_A$ of any analytic set $A \subset M$ is coherent.*

It of course follows from Oka and Cartan's theorems that the sheaf $\mathcal{O}_A$ is coherent.

Loosely speaking, we obtain a complex space by gluing several analytic sets together. Gluing sets is done via morphisms, so to make this notion precise we need a suitable notion of a morphism of analytic sets:

Definition 2.1.5. Let $A \subset M$ and $B \subset N$ be analytic subsets of analytic manifolds M and N. A *morphism* from A to B is a map $F : A \to B$ such that for every x in A there exists a neighborhood U of x and a holomorphic map $\tilde{F} : U \to N$ such that $\tilde{F}|_{U \cap A} = F|_{U \cap A}$. At every point $x \in A$ the map F induces a ring morphism

$$F_x^* : \mathcal{O}_{B,F(x)} \to \mathcal{O}_{A,x}, \quad g \mapsto g \circ F_x$$

called the *comorphism* of F.

Definition 2.1.6. Let X be a locally compact Hausdorff space, countable at infinity (note: also known as σ-compactness), with a sheaf of continuous functions $\mathcal{O}_X$. Then X is said to be a *complex space* if there exists an open covering (U_λ) of X along with homeomorphisms $F_\lambda : U_\lambda \to A_\lambda$ onto analytic subsets $A_\lambda \subset \mathbb{C}^{n_\lambda}$, such that the comorphisms $F_\lambda^* : \mathcal{O}_{A_\lambda} \to \mathcal{O}_{X|U_\lambda}$ are sheaf isomorphisms. The sheaf $\mathcal{O}_X$ is called the *structure sheaf* of X.

Note that both analytic manifolds and analytic sets are complex spaces, and that the structure sheaf of a complex space is coherent. Now, a complex space X is locally isomorphic to an analytic set, so we have well defined notions of holomorphic functions on X, analytic subsets of X, morphisms of complex spaces, and so forth.

Remark — In geometric terms, what we have defined here is a local ringed space which is locally isomorphic to ringed spaces of the type $(A, \mathcal{O}_A)$ where A is an analytic set. Although introducing the notion of a ringed space would let us give a more compact definition of a complex space and save us some time later in defining algebraic varieties, we will use the more explicit definition above in the next chapter and would have to give it anyway.

We finish this section with a very important definition:

Definition 2.1.7. Let X be a complex space with structure sheaf $\mathcal{O}_X$. An *analytic sheaf* is a sheaf of $\mathcal{O}_X$-modules.

2.2 Algebraic varieties

Put very briefly, we are now going to repeat the last section, but with polynomials and rational maps in the place of holomorphic maps and to define algebraic varieties.

There are some differences between the algebraic and analytic cases though, as for example it will be natural to consider a different underlying topology on an algebraic set in place of the usual one on an analytic set. For motivation, we recall the small spoiler that rational functions will play a role in our construction, and consider that any rational function on an open set U in $\mathbb{C}^r$ will extend to a larger open set W unless there exists a non-zero polynomial which has no zeros in U but has zeros in W. In a sense we want to work with open sets which are an analog of domains of holomorphy for rational functions. This leads us to define the following topology on $\mathbb{C}^r$:

Definition 2.2.1. A set $A \subset \mathbb{C}^r$ is said to be an *Zariski-closed* (or Z-closed) if there exist polynomials $p_1, \ldots, p_N$ on $\mathbb{C}^r$ such that

$$A = \{z \in \mathbb{C}^r \,|\, p_1(z) = \ldots = p_N(z) = 0\}$$

A *Zariski-open* (or Z-open) set will be the complement of a Z-closed set. The topology defined by these open sets is called the *Zariski-topology* on $\mathbb{C}^r$.

Proving that the collection of Zariski-open sets does indeed form a topology is not hard and is done by considering ideals in the polynomial ring $\mathbb{C}[X_1, \ldots, X_r]$, as they correspond to closed sets. Note also that the Zariski-topology is coarser than the usual topology on $\mathbb{C}^r$, i.e. every Zariski-open set is open in the usual sense.

Definition 2.2.2. Let U be a Zariski-open set. A function R on U is *regular* if it is continuous and there exist polynomials P and Q such that Q is non-zero on U and $R = P/Q$.

If we give $\mathbb{C}$ and $\mathbb{C}^r$ the Zariski-topology, then the regular functions on $\mathbb{C}^r$ are continuous and form a sheaf $\mathcal{O}_{\mathbb{C}^r}$ over $\mathbb{C}^r$.

As an analog of the analytic sets of the last section we define:

Definition 2.2.3. A subset Y of $\mathbb{C}^r$ is said to be *locally closed* if $Y = U \cap F$, where U is Z-open and F is Z-closed set. A function $R : Y \to \mathbb{C}$ is called *regular* if there exists a Z-open neighborhood V of Y and a regular function $\tilde{R}$ on V such that $\tilde{R}_{|Y} = R$. The sheaf of regular functions on Y is denoted by $\mathcal{O}_Y$.

If $Y = U \cap F$ is locally closed, then we can form a sheaf of ideals $\mathcal{I}_F \subset \mathbb{C}[X_1, \ldots, X_r]$, called the *ideal sheaf* of Y. The sections of $\mathcal{I}_F$ are the polynomials which are zero on F. Similarly to the analytic case we have that $\mathcal{O}_Y = (\mathcal{O}_{\mathbb{C}^r}/\mathcal{I}_F)_{|Y}$.

The following proposition is proved in [Ser55], it follows from the proofs of Oka and Cartan's theorems:

Proposition 2.2.4. *The sheaves $\mathcal{O}_{\mathbb{C}^r}$, $\mathcal{I}_F$ and $\mathcal{O}_Y$ are coherent.*

In another analog with the last section we define maps between locally closed sets:

Definition 2.2.5. Let $U \subset \mathbb{C}^r$ and $V \subset \mathbb{C}^s$ be locally closed. A map $\varphi : U \to V$ is called a *regular map* is φ is continuous and for any $x \in U$ there is a neighborhood $W \subset \mathbb{C}^r$ of x and a function $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_s) : W \to \mathbb{C}^s$ such that each of the $\tilde{\varphi}_j$ is a regular function and $\tilde{\varphi}_{U \cap W} = \varphi_{U \cap W}$.

We can now define the algebraic version of a complex space. As before we obtain an algebraic variety by gluing together several locally closed sets, but there are slightly different conditions on the resulting topological space than before.

Definition 2.2.6. An *algebraic variety* is a topological space X with a sheaf of rings $\mathcal{O}_X$ such that the diagonal in $X \times X$ is closed, and there exists a finite covering $U_1, \ldots, U_N$ of X and homeomorphisms $F_j : U_j \to V_j$, where V_j is a locally closed set in $\mathbb{C}^{r_j}$, such that the comorphism $F_j^* : \mathcal{O}_{V_j} \to \mathcal{O}_{X|U_j}$ is a sheaf isomorphism. The sheaf $\mathcal{O}_X$ is called the *structure sheaf* of X.

Remark — Please note that the condition that the diagonal is closed Δ in $X \times X$ does *not* imply that X is Hausdorff, because the Zariski topology on $X \times X$ is in general not the product topology. That Δ be closed is imposed here to exclude certain pathological examples from the class of algebraic varieties.

Definition 2.2.7. Let X be an algebraic space with a structure sheaf $\mathcal{O}_X$. An *algebraic sheaf* over X is a sheaf of $\mathcal{O}_X$-modules.

We will be concerned with a particular type of algebraic varieties called projective varieties. As the name implies they have something to do with the projective space $\mathbb{P}^n$, which is defined as the quotient of $\mathbb{C}^{n+1} \backslash \{0\}$ by the action of the multiplicative group $\mathbb{C}^*$. There is thus a projection $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ and we give $\mathbb{P}^n$ the quotient topology where $\mathbb{C}^{n+1} \setminus \{0\}$ has the Zariski-topology. One can show that this topology is the same one as the one defined by setting a closed set equal to the zero set of a finite number of homogeneous polynomials on $\mathbb{P}^n$.

We define a sheaf $\mathcal{O}_{\mathbb{P}^n}$ over $\mathbb{P}^n$, which we call the *sheaf of regular functions* on $\mathbb{P}^n$, by setting

$$\mathcal{O}_{\mathbb{P}^n}(U) := \{f \in \mathcal{O}_{\mathbb{C}^{n+1}}(\pi^{-1}(U)) |\, f(\lambda x) = f(x) \text{ for all } \lambda \in \mathbb{C}^*\}$$

for every open set $U \subset P^n$, and we take the obvious restriction maps. A regular function on a subset of $\mathbb{P}^n$ may be thought of as a rational function P/Q where Q is non zero on U and P and Q are homogeneous polynomials of the same degree.

The topological space $\mathbb{P}^n$ with the sheaf $\mathcal{O}_{\mathbb{P}^n}$ is the prototype of a projective variety, the terminology being justified by the following proposition proved in [Ser55]:

Proposition 2.2.8. *The space $\mathbb{P}^n$ with the sheaf $\mathcal{O}_{\mathbb{P}^n}$ is an algebraic variety.*

The objects we will work with in GAGA are *projective varieties,* which are defined as closed subvarieties of $\mathbb{P}^n$.

2.3 Analytification

In the previous section we defined the Zariski-topology on $\mathbb{C}^n$. We can compare this topology with the usual one:

Lemma 2.3.1.

a) The Zariski topology on $\mathbb{C}^n$ is coarser than the usual one.

b) Every locally Z-closed subset is analytic.

c) If U and U' are locally Z-closed and $f : U \to U'$ is a regular map, then f is an analytic map.

d) If the map f in c) is a regular isomorphism, then it is an analytic isomorphism.

Proof: Statement a) is obvious because any Z-closed set is closed in the usual topology and b) is true because polynomials are holomorphic. The component functions of a regular map are rational functions, which are holomorphic, so c) is immediate, and d) follows if we apply c) to the inverse f^{-1}. □

Given an algebraic variety X, we are now going to use this lemma to define the structure of an analytic space on X. In order to not confuse the two topologies we will obtain on X, we will refer to an open set in the original topology as a Z-open set, and refer to an open set in the new topology simply as an open set.

Proposition 2.3.2. *Let X be an algebraic variety. There exists a unique analytic structure on X such that every Zariski-open set $U \subset X$ will be open, and such that any regular morphism $\varphi : U \to V$ from $U \subset X$ to a locally Z-closed $V \subset \mathbb{C}^n$ will be analytic.*

Proof: Let U_j be an open covering of X such that there are homeomorphisms $\varphi_j : U_j \to V_j$, where V_j is a locally closed set in $\mathbb{C}^{r_j}$. We give V_j the structure of an analytic set via the lemma, that is the usual topology and a sheaf $\mathcal{O}_{V_j}$ of holomorphic functions, and transfer that structure to U_j with the map φ_j^{-1}. More precisely, we define a basis for a topology on X by the sets $\varphi_j^{-1}(W)$ where $W \subset V_j$ is open in the usual topology on V_j, we define a sheaf $\mathcal{O}_X^{an}$ locally with the inverse images $\varphi_j^{-1}\mathcal{O}_{V_j}$, and glue together via lemma 1.1.8.

Denote the space X with this new structure by X^{an}. We note that X^{an} is indeed a topological space with a sheaf of rings which is locally isomorphic to an analytic set and its sheaf of holomorphic functions. As the analytic sets V_j are locally compact and Hausdorff, then so is X^{an}, and because the V_j are countable at infinity and X^{an} is covered by a finite number of sets isomorphic to V_j, then X^{an} is countable at infinity. Thus X^{an} is a complex space.

If U is Z-open in X and $\varphi : U \to V$ is a morphism onto a locally Z-closed V in $\mathbb{C}^n$, then U is the union of the open $U \cap U_j$ and therefore open in X^{an}. The map φ is regular, so by construction it will belong to $\mathcal{O}_X^{an}$ and be analytic.

Any analytic structure defined on X via other homeomorphisms $\psi_j : U'_k \to V'_k$ will now be homeomorphic to the one defined here via comparison on the intersections $U_j \cap U'_k$. □

Remark — If X is an algebraic variety then we have a continuous map $\iota : X^{an} \to X$, which set-theoretically is just the identity map. Since regular functions are holomorphic in the usual topology, $\iota^{-1}\mathcal{O}_{alg}$ is a subring of $\mathcal{O}_{an}$, which then has the structure of a $\iota^{-1}\mathcal{O}_{alg}$-module.

Remark — Now, given an algebraic sheaf $\mathcal{F}$ on an algebraic variety X, we can apply the modified inverse image operation from definition 1.3.4 to $\mathcal{F}$ and obtain an analytic sheaf $\iota^*\mathcal{F}$ on X^{an}. We will modify our notation slightly and write $\mathcal{F}^{an}$ instead of $\iota^*\mathcal{F}$. We call $\mathcal{F}^{an}$ the analytic sheaf associated to $\mathcal{F}$, or the *analytification* of $\mathcal{F}$.

Remark — Let us recall some properties of this operation from section 1.3: We have a natural morphism $\alpha : \mathcal{F} \to \mathcal{F}^{an}$ for any algebraic sheaf $\mathcal{F}$, and in particular a natural morphism $\mathcal{O}_{alg} \to (\mathcal{O}_{alg})^{an} = \mathcal{O}_{an}$. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of algebraic sheaves, then φ gives rise to a canonical morphism $\varphi^{an} : \mathcal{F}^{an} \to \mathcal{G}^{an}$ of analytic sheaves. Also, if $\psi : \mathcal{G} \to \mathcal{H}$ is another morphism of algebraic sheaves, then $(\psi \circ \varphi)^{an} = \psi^{an} \circ \varphi^{an}$. And finally, if $\mathcal{F}$ is a coherent algebraic sheaf, then $\mathcal{F}^{an}$ is a coherent analytic sheaf.

A few properties of the analytification operation are obtained as consequences of the fact that $\mathcal{O}_{alg}$ and $\mathcal{O}_{an}$ form a faithful flat pair of rings. For our purposes this means that for any point x and any exact sequence $E \to F \to G$ of $\mathcal{O}_{alg,x}$-modules the sequence

$$E \otimes_{\mathcal{O}_{alg,x}} \mathcal{O}_{an,x} \to F \otimes_{\mathcal{O}_{alg,x}} \mathcal{O}_{an,x} \to G \otimes_{\mathcal{O}_{alg,x}} \mathcal{O}_{an,x}$$

is exact, and that if E is a non-zero $\mathcal{O}_{alg,x}$-module then $E \otimes_{\mathcal{O}_{alg,x}} \mathcal{O}_{an,x}$ is non-zero. The first property says that $\mathcal{O}_{alg}$ and $\mathcal{O}_{an}$ form a *flat* pair of rings, and the second says that the flat pair is *faithful.* That $\mathcal{O}_{alg}$ and $\mathcal{O}_{an}$ form a faithful flat pair of rings is proved by showing that they are noetherian rings with the same completions. We will not go into detail on how these algebraic properties are proved and used in the demonstration of the next three propositions, and instead refer the reader to [Ser56].

Proposition 2.3.3. *The canonical map $\mathcal{O}_{alg} \to \mathcal{O}_{an}$ is injective.*

Because of the his proposition we will from now on identify $\mathcal{O}_{alg}$ with its image in $\mathcal{O}_{an}$. From the flatness property we get:

Corollary 2.3.4. *Let $\mathcal{F} \to \mathcal{S} \to \mathcal{G}$ be an exact sequence of algebraic sheaves. Then $\mathcal{F}^{an} \to \mathcal{S}^{an} \to \mathcal{G}^{an}$ is exact.*

The sheaves of homomorphisms of algebraic and analytic sheaves satisfy a particularly nice property under analytification:

Proposition 2.3.5. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent algebraic sheaves, then there is a canonical isomorphism*

$$Hom_{\mathcal{O}_{alg}}(\mathcal{F}, \mathcal{G})^{an} = Hom_{\mathcal{O}_{an}}(\mathcal{F}^{an}, \mathcal{G}^{an})$$

Finally we have a proposition which tells us that it doesn't matter in which order we extend or analytify a sheaf on a closed subvariety. It will be used, along with the results of section 1.3 to reduce GAGA to an easier special case.

Proposition 2.3.6. *Let $Y \subset X$ be a closed subvariety of an algebraic variety X, and let $\mathcal{F}$ be an algebraic sheaf on Y. If $\mathcal{F}^X$ is the trivial extension of $\mathcal{F}$ to all of X, then*

$$(\mathcal{F}^{an})^{X^{an}} = (\mathcal{F}^X)^{an}$$

Chapter 3

The Cartan-Serre theorem

For the proof of GAGA we will need a result due to Cartan and Serre on the finiteness of the dimension of the cohomology groups of a coherent analytic sheaf on a compact analytic space. Establishing the result itself is fairly painless once we've developed the necessary machinery, and most of the chapter will be devoted to its development.

Let's start by stating the theorem and outlining our plan of attack, to give motivation for the development of much of the results in the chapter. Our goal is to prove the following result:

Theorem. (Cartan-Serre) *Let $\mathcal{F}$ be a coherent analytic sheaf on a compact analytic space X. Then $H^k(X, \mathcal{F})$ is finite dimensional for all $k \geq 0$.*

We intend to prove this theorem with an application of a theorem of Schwartz from functional analysis:

Theorem. (Schwartz) *Let $(E^\bullet, d)$ and $(F^\bullet, \delta)$ be complexes of Fréchet spaces with continuous differentials, and let $\rho^\bullet : E^\bullet \to F^\bullet$ be a continuous complex morphism. If ρ^q is compact and $H^q(\rho^\bullet) : H^q(E^\bullet) \to H^q(F^\bullet)$ is surjective, then $H^q(F^\bullet)$ is a Hausdorff finite dimensional vector space.*

To apply Schwartz's theorem we need to find suitable complexes and a morphism between them. Natural candidates are the complex of Čech cochains of the sheaf $\mathcal{F}$, or more precisely two Čech complexes defined by acyclic coverings $\mathcal{U}$ and $\mathcal{U}'$ where one covering is finer than the other, and the restriction morphism between the complexes.

We thus have several objectives: we need to establish Schwartz's theorem, produce a covering of X, prove that this covering is acyclic with respect to $\mathcal{F}$, define a Fréchet space structure on the Čech complex defined by this covering, and prove that if we take a finer covering with the same properties then the restriction morphism between the two complexes will be compact.

3.1 Schwartz's theorem

We will need some results on the perturbation of linear operators.

Definition 3.1.1. Let E and F be Fréchet spaces and $g : E \to F$ be a continuous linear operator.

a) g is *compact* if there exists a neighborhood U of 0 in E such that the image $\overline{g(U)}$ is compact in F.

b) g is a *(quasi-) monomorphism* if $g(E)$ is closed in F and g is injective (resp. if $\operatorname{Ker} g$ is finite dimensional).

c) g is an *(quasi-) epimorphism* if g is surjective (resp. if $g(E)$ is finite codimensional).

Remark — The notions of mono- and epimorphisms are usually defined on more general types of topological vector spaces where Banach's theorem doesn't always apply. There we insist, for example, that an epimorphism be a surjective open map. We will only make use of Fréchet spaces, and adjust our definitions of mono- and epimorphisms accordingly.

Theorem 3.1.2. *Let $h : E \to F$ be a compact linear operator.*

a) If $g : E \to F$ is a quasi-monomorphism, then $g + h$ is a quasi-monomorphism.

b) If $g : E \to F$ is a quasi-epimorphism, then $g+h$ is a quasi-epimorphism.

A proof of this theorem can be found in chapter 9 in [Dem07], where it is in fact proved for more general Hausdorff locally convex topological vector spaces. We will only use it to establish Schwartz's theorem.

Theorem 3.1.3. (Schwartz) *Let $(E^\bullet, d)$ and $(F^\bullet, \delta)$ be complexes of Fréchet spaces with continuous differential, and let $\rho^\bullet : E^\bullet \to F^\bullet$ be a continuous complex morphism. If ρ^q is compact and $H^q(\rho^\bullet) : H^q(E^\bullet) \to H^q(F^\bullet)$ is surjective, then $H^q(F^\bullet)$ is a Hausdorff finite dimensional vector space.*

Proof: Begin by defining the operators

$$\begin{aligned} g, h &: Z^q(E^\bullet) \oplus F^{q-1} \to Z^q(F^\bullet), \\ g(x \oplus y) &= \rho^q(x) + \delta^{q-1}(y), \\ h(x \oplus y) &= -\rho^q(x) \end{aligned}$$

As $Z^q(E^\bullet) \subset E^q$ and $Z^q(F^\bullet) \subset F^q$ are the kernels of continuous morphisms, they are closed, and thus Fréchet spaces. The operator h is compact because ρ^q is compact. Now, let $z \in Z^q(F^\bullet)$ and find $[e] \in H^q(E^\bullet)$ such that $H^q(\rho)([e]) = [z]$. Then there exists a y in F^{q-1} such that $\rho(e) = z + \delta^{q-1}(y)$, and we find that $g(e \oplus -y) = z$, so g is an epimorphism. We then find that $f = g + h = \delta^{q-1}$ is a quasi-epimorphism, so $B^q(F^\bullet)$ is closed and finite codimensional in $Z^q(F^\bullet)$. Thus $H^q(F^\bullet)$ is Hausdorff and finite dimensional.□

3.2 Pseudoconvex neighborhoods

The open covering of X we will take will be one of strongly pseudoconvex neighborhoods, or equivalently, of Stein neighborhoods. We choose pseudoconvex sets because they are closed under finite intersections, certain cohomology groups disappear on them, and there are a lot of them. We could also have gotten by with, for example, polynomial polyhedra or polynomially convex sets, which are both simple special cases of pseudoconvex sets.

Let X be a complex manifold, x a point in X and $(z_1, \dots, z_n)$ local coordinates on a neighborhood of X. For any $u \in \mathcal{C}^2(X)$ we define a hermitian form on $T_{X,x}$ by

$$Hu_x := \sum_{1 \le j,k \le n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(x)\, \mathrm{d}z_j \otimes \mathrm{d}\bar{z}_k,$$

and we call Hu_x the *complex Hessian* of u at x.

Now let Y be another complex manifold, $f : X \to Y$ a holomorphic mapping and $v \in \mathcal{C}^2(Y)$. If x is a point in X we calculate that for $\xi \in T_{X,x}$

$$H(v \circ f)_x(\xi) = \sum_{l,m,j,k} \frac{\partial^2 (v \circ f)}{\partial z_l \partial \bar{z}_m}(x) \frac{\partial f_m(x)}{\partial z_j} \xi_j \overline{\frac{\partial f_m(x)}{\partial z_k}} \xi_k = Hv_{f(x)}(f'(x) \cdot \xi)$$

so the definition of Hu_x does not depend on the choice of coordinates $(z_1, \dots, z_n)$ on X. We also see that if $Hv_{f(x)}$ is (semi-)positive, then so is $H(v \circ f)_x$. We now define:

Definition 3.2.1. Let X be an analytic manifold. A function $\varphi \in \mathcal{C}^2(X)$ is *(strictly) plurisubharmonic* if its complex Hessian $H\varphi$ is semi-positive (resp. positive) on X.

Remark — (i) By the discussion above we see that if X and Y are complex manifolds, $f : X \to Y$ is holomorphic and φ is (strictly) plurisubharmonic on Y, then $\varphi \circ f$ is (strictly) plurisubharmonic on X. In particular, if $X \subset Y$ then $\varphi|_X$ is plurisubharmonic on X.

(ii) Note that $H(u+v)_x = Hu_x + Hv_x$, so the sum of plurisubharmonic functions is plurisubharmonic, and the sum is strictly plurisubharmonic as soon as one of the functions being summed is strictly plurisubharmonic.

(iii) This is not the usual definition of a plurisubharmonic function, which is usually taken to be an upper semicontinuous function with values in $[-\infty, +\infty[$ which is subharmonic when restricted to any complex line. Our definition is equivalent to the usual one when the function in question is at least twice continuously differentiable. However we are only going to use plurisubharmonic functions to define pseudoconvex sets, and our restriction poses no problems for that purpose.

Definition 3.2.2. Let X be a complex space, let x be a point of X, and let φ be a function defined on X. We say that φ is (strongly) plurisubharmonic at x if there exists a neighborhood U of x, a biholomorphic $f : U \to A$ onto an analytic set $A \subset \Omega \subset \mathbb{C}^N$ and a (strongly) plurisubharmonic function ψ on Ω such that $\varphi \circ f = \psi|_A$. The function φ is *(strongly) plurisubharmonic* on X if it is (strongly) plurisubharmonic at every point of X.

We can show that the definition of a plurisubharmonic function is independent of the embedding of U in $\mathbb{C}^N$.

Definition 3.2.3. A function $\psi : X \to [-\infty, \infty[$ on a topological space X is called an *exhaustion function* if for any $c \in \mathbb{R}$ the sublevel set

$$X_c := \{x \in X |\, \psi(x) < c\}$$

is relatively compact in X.

Definition 3.2.4. Let X be a complex space. Then X is said to be *strongly pseudoconvex* if there exists a strongly plurisubharmonic exhaustion function on X.

Example 3.2.5. (i) The function $\varphi(z) := -\log(r^2 - |z-x|^2)$ is strictly plurisubharmonic on any open ball $B = B(x,r) = \{z \in \mathbb{C}^n ||z-x| < r\}$. It is clearly an exhaustion function on B, so B is strongly pseudoconvex.

(ii) Now set $\psi(z) = -\log(1+|z|^2)$. Then ψ is a strictly plurisubharmonic exhaustion function on $\mathbb{C}^n$, so $\mathbb{C}^n$ is strongly pseudoconvex.

Proposition 3.2.6. *Let X be a strongly pseudoconvex complex space.*

a) If A is a closed analytic subset of X, then A is strongly pseudoconvex.

b) If $U_1, \ldots, U_N$ are strongly pseudoconvex subspaces of X such that $U := U_1 \cap \ldots \cap U_N$ is a subspace of X, then U is strongly pseudoconvex.

Proof: Let φ be a strongly plurisubharmonic exhaustion function on X.

a) As A is closed, then $\varphi|_A$ is an exhaustion function on A.

b) Note that X^N is strongly pseudoconvex because $\psi(x_1, \ldots, x_n) = \varphi(x_1) + \ldots + \varphi(x_n)$ is an exhaustion function on X^N. In the same way the product $\prod U_j$ is strongly pseudoconvex. If Δ is the diagonal of X^N, then Δ is closed, so $U = \Delta \cap \prod U_j$ is a closed submanifold of $\prod U_j$ and therefore strongly pseudoconvex. □

Proposition 3.2.7. *If X is a complex space, then every point x of X has a fundamental family of strongly pseudoconvex neighborhoods, i.e. for every open neighborhood W of x there exists a strongly pseudoconvex neighborhood U of x such that $U \subset W$.*

Proof: We reduce to the case where X is an analytic set $A \subset \Omega \subset \mathbb{C}^N$. For any neighborhood W of x there exists a ball $B(x,r) \subset \Omega$ such that $U := B(x,r) \cap A \subset W$. But $B(x,r)$ is strongly pseudoconvex and A is closed, so U is strongly pseudoconvex. □

The reason why we introduce strongly pseudoconvex neighborhoods lies in the following theorem:

Theorem 3.2.8. *Let X be a strongly pseudoconvex analytic manifold and let E be a holomorphic vector bundle on X. Then $H^{p,q}_{\bar\partial}(X,E) = 0$ for any $p \geq 0$ and any $q > 0$.*

Remark — (i) A nice proof of this theorem is by way of L^2 estimates on the sections of E. We will not give the proof here, as just giving the necessary definitions and preparatory results to begin the proof would be material enough for a separate chapter, and instead refer the interested reader to [Dem07].

(ii) Siu has given a proof of the stronger Cartan B theorem for strongly pseudoconvex sets which uses only (relatively) elementary results. Details may be found in [Siu68].

(iii) It is worth mentioning that for our purposes we only need to have a fundamental collection of sets U around every point of a complex space which are closed under finite intersections and have the property that $H^{0,q}_{\bar\partial}(U,\mathbb{C}) = 0$ for every $q > 0$. As we said earlier, an alternative collection of such sets is formed by a generalization of polydisks, or polynomial polyhedra. They are trivially closed under intersections, and annihilating their cohomology groups consists of pulling oneself up by ones bootstraps from the easy case of polydisks. The motivated reader can find the relevant details in [GR65], from which one can extend the same result to polynomially convex sets.

3.3 A topology on $\mathcal{F}$

Let X be a complex space and let $\mathcal{F}$ be a coherent analytic sheaf on X. By the previous section there exists a fundamental family of strongly pseudoconvex sets around every point of X.

Remark — Our goal in the first part of this section is to prove theorem 3.3.3, which says that there exists a fundamental family of neighborhoods around any point x such that all the higher cohomology groups of a coherent analytic sheaf vanish on those neighborhoods. As our neighborhoods are strongly pseudoconvex, this is a consequence of the Cartan B theorem. The point of this development is thus to see that we can get by without Cartan's theorem, any readers familiar with it can safely jump to the construction of the topology on $\mathcal{F}$ in the end of the section.

Proposition 3.3.1. *If M is a strongly pseudoconvex complex manifold then*

$$H^k(M, \mathcal{O}^p) = 0$$

for all $p \geq 1, k \geq 1$.

Proof: By the Dolbeault isomorphism theorem and theorem 3.2.8 we have

$$H^k(M, \mathcal{O}^p) = (H^k(M, \mathcal{O}))^p = (H^{0,k}_{\bar{\partial}}(M, \mathbb{C}))^p = 0, \quad k \geq 1. \qquad \square$$

Now, for $x \in X$ there exists an open neighborhood $V \subset X$ of x and a homeomorphism $F : V \to A \subset \Omega \subset \mathbb{C}^N$, where A is an analytic set and $F^* : \mathcal{O}_{|A} \to \mathcal{O}_{X|V}$ is a sheaf isomorphism. Let $i : A \hookrightarrow \Omega$ be the inclusion. Then $i_*\mathcal{F}$ is a coherent $\mathcal{O}_\Omega$-module supported on A. By theorem 1.2.10 there exists an exact sequence

$$\mathcal{O}^{p_{2N}} \to \mathcal{O}^{p_{2N-1}} \to \ldots \to \mathcal{O}^{p_1} \to \mathcal{O}^{p_0} \to i_*\mathcal{F} \to 0 \tag{3.1}$$

on a neighborhood $U \subset \Omega$. In this case we shall say that $A \subset \Omega$ is a *$\mathcal{F}$-distinguished analytic set.* After replacing Ω by U and A by $A \cap U$, we may assume that the exact sequence (3.1) exists on all of Ω. We have proven:

Proposition 3.3.2. *Let $\mathcal{F}$ be a coherent analytic sheaf on a complex space X. Then every x in X is contained in some $\mathcal{F}$-distinguished analytic set A.*

Theorem 3.3.3. *Let $A \subset \Omega$ be a $\mathcal{F}$-distinguished analytic set. If $U \subset \Omega$ is strongly pseudoconvex and $V = A \cap U$, then $H^k(V, \mathcal{F}) = 0$ for all $k \geq 1$.*

Proof: We denote by $\mathcal{Z}^l$ the kernel of the map $\mathcal{O}^{p_l} \to \mathcal{O}^{p_{l-1}}$ for $l \geq 1$ and by $\mathcal{Z}^0$ the kernel of $\mathcal{O}^{p_0} \to i_*\mathcal{F}$. There are exact sequences

$$0 \to \mathcal{Z}^0 \to \mathcal{O}^{p_0} \to i_*\mathcal{F} \to 0$$
$$0 \to \mathcal{Z}^l \to \mathcal{O}^{p_l} \to \mathcal{Z}^{l-1} \to 0$$

for $1 \leq l \leq 2N$. By proposition 3.3.1 we have that $H^k(U, \mathcal{O}^p) = 0$ for all $k \geq 1$, so we get a string of isomorphisms

$$H^k(V, \mathcal{F}) \simeq H^k(U, i_*\mathcal{F}) \simeq H^{k+1}(U, \mathcal{Z}^0) \simeq \ldots \simeq H^{k+2N+1}(U, \mathcal{Z}^{2N}),$$

and the last group vanishes because $\operatorname{topdim} V \leq \dim_{\mathbb{R}} V = 2N$. $\square$

(3.3.4) Construction. Let $A \subset \Omega$ be a $\mathcal{F}$-distinguished analytic set, let $U \subset \Omega$ be strongly pseudoconvex, and set $V = A \cap U$. By the proof of the above theorem we have $H^1(U, \mathcal{Z}^0) = 0$, so we have an exact sequence

$$0 \to \mathcal{Z}^0(V) \to \mathcal{O}^p(V) \to \mathcal{F}(U) \to 0,$$

and we have a Fréchet space structure on $\mathcal{O}^p(V)$. We will now show that $\mathcal{Z}^0(V)$ is closed in $\mathcal{O}^p(V)$, so $\mathcal{Z}^0(V)$ is a Fréchet space, and we can give $\mathcal{F}(U)$ the Fréchet structure of the quotient space:

Let $f_m \in \mathcal{Z}^0(V)$ be a sequence converging uniformly on compacts in V to a limit f in $\mathcal{O}^p(V)$. Then the germs $(f_m)_x$ converge to f_x with respect to the Krull topology defined on $\mathcal{O}^p(V)$ (see lemma A.2.1). But by theorem A.2.2 we know that $\mathcal{Z}^0_x$ is closed in $\mathcal{O}^p_x$ with respect to this topology, so $f_x \in \mathcal{Z}^0_x$ for all x and thus $f \in \mathcal{Z}^0(V)$, i.e. $\mathcal{Z}^0(V)$ is closed in $\mathcal{O}^p(V)$.

Proposition 3.3.5. *The topology on $\mathcal{F}(U)$ is independent of the choices made above.*

A proof of this proposition may be found in chapter 9 of [Dem07].

3.4 The Cartan-Serre theorem

For each x in X we let $A \subset \Omega$ be a $\mathcal{F}$-distinguished patch that contains x, and $V \subset \Omega$ a strongly pseudoconvex neighborhood of x. We then have a covering of X by neighborhoods $U = V \cap A$ which are restrictions of strongly pseudoconvex sets to X, and we can extract a countable covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of such sets. Note that finite intersections

$$U_{\alpha_0,\dots,\alpha_q} := \bigcap_{j=0}^{q} U_{\alpha_j}$$

are again restrictions of strongly pseudoconvex sets to X which are contained in $\mathcal{F}$-distinguished analytic sets. By the results in the last section, $\mathcal{U}$ is an acyclic covering of X and Leray's theorem tells us that $\check{H}^k(\mathcal{U}, \mathcal{F}) \simeq H^k(X, \mathcal{F})$ for all $k \geq 0$.

We now consider the product topology on the spaces of Čech cochains

$$\mathcal{C}^q(\mathcal{U}, \mathcal{F}) := \bigoplus_{(\alpha_0,\dots,\alpha_q) \in I^{q+1}} \mathcal{F}(U_{\alpha_0,\dots,\alpha_q}).$$

As the restriction maps $\rho_{UV} : \mathcal{O}^p(V) \to \mathcal{O}^p(U)$ are continuous, so are the restriction maps on $\mathcal{F}$, and thus the restriction maps on $\mathcal{C}^q(\mathcal{U}, \mathcal{F})$ are continuous as well. It follows that the Čech-differentials $\delta^q : \mathcal{C}^q(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{q+1}(\mathcal{U}, \mathcal{F})$

are continuous. We give the cohomology groups $\check{H}^k(\mathcal{U}, \mathcal{F})$ the topological structure of the quotient space, and we see that the maps induced by the restriction maps and the Čech-differentials are continuous.

Lemma 3.4.1. *Let U and U' be the restrictions of strongly pseudoconvex neighborhoods contained in a $\mathcal{F}$-distinguished analytic set and suppose that U' is relatively compact in U. Then the restriction operator $\rho_{U'U} : \mathcal{F}(U) \to \mathcal{F}(U')$ is compact.*

Proof: Let $U = A \cap V$ and $U' = A \cap V'$ where V and V' are strongly pseudoconvex sets and V' is relatively compact in V. The restriction operator $\mathcal{O}^p(V) \to \mathcal{O}^p(V')$ is compact by Montel's theorem, so $\rho_{U'U} : \mathcal{F}(U) \to \mathcal{F}(U')$ is also compact. □

Theorem 3.4.2. (Cartan-Serre) *Let $\mathcal{F}$ be a coherent analytic sheaf on a compact complex space X. Then $H^k(X, \mathcal{F})$ is finite dimensional for all $k \geq 0$.*

Proof: By the preceding discussion we can find a suitable covering $\mathcal{U}$ of X and give the corresponding Čech complex and cohomology groups a topological structure. Because X is compact we can take the $\mathcal{U}$ to be finite, and we can find another covering of strongly pseudoconvex sets $\mathcal{U}' = (U'_\alpha)$ such that $U'_\alpha \subset\subset U_\alpha$ for all α.

From the above lemma we conclude that the restriction morphism $\rho^\bullet : \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^\bullet(\mathcal{U}', \mathcal{F})$ is compact. It induces a morphism $H^q(\rho^\bullet) : \check{H}^q(\mathcal{U}, \mathcal{F}) \to \check{H}^q(\mathcal{U}', \mathcal{F})$, which is surjective because both cohomology groups are isomorphic to $H^k(X, \mathcal{F})$ via isomorphisms which fit into a commutative diagram with the restriction morphism. The result now follows from Schwartz's theorem. □

Chapter 4

GAGA

It is worth taking a moment to consider the information we have gathered so far. In chapter 2 we saw that given a coherent algebraic sheaf $\mathcal{F}$ on an algebraic variety X, we can fabricate an analytic sheaf $\mathcal{F}^{an}$ on the complex space X^{an}. We also have a natural sheaf morphism $\alpha : \mathcal{F} \to \mathcal{F}^{an}$, which induces a morphism $\alpha^{\bullet}$ of the complex of sheaf cohomology groups. Several questions now present themselves, such as:

Does every analytic sheaf on X^{an} arise in this way? If so, then does it arise from a unique algebraic sheaf? What do we know about the morphisms α^k? Are they injective or surjective or isomorphisms?

As it turns out it is too optimistic to hope that the answers to these questions are positive in general:

Example 4.0.3. There exist analytic sheaves which do not rise as analytifications of algebraic sheaves: Consider the ideal sheaf $\mathcal{I}_A$ of the analytic set

$$A = \{z \in \mathbb{C} \mid \sin z = 0\}$$

If there was an algebraic sheaf $\mathcal{F}$ over $\mathbb{C}$ such that $\mathcal{F}^{an} = \mathcal{I}_A$ we would find that $A = \operatorname{supp} \mathcal{I}_A = \operatorname{supp} \mathcal{F}$, so A would be the support of a locally finitely generated sheaf and thus closed in the Zariski topology. But A is a countably infinite set, and thus not the zero set of any finite number of polynomials.

Example 4.0.4. Even when an analytic sheaf is the analytification of an algebraic sheaf, the algebraic sheaf is not necessarily unique. We don't have space to give their construction, but Shavarevich defines in [Sha94] two algebraic varieties X and Y which are not isomorphic, but whose analytifications are isomorphic. By definition the coherent analytic sheaf $\mathcal{O}_{X^{an}} \simeq \mathcal{O}_{Y^{an}}$ rises from $\mathcal{O}_X$ and $\mathcal{O}_Y$, but these cannot be isomorphic as then the underlying algebraic varieties would be isomorphic as well.

Example 4.0.5. The induced maps are not always isomorphisms: There are entire holomorphic functions on $\mathbb{C}$ which are not polynomials, or put another way, while $\alpha^0 : H^0(\mathbb{C}, \mathcal{O}_{alg}) \to H^0(\mathbb{C}, \mathcal{O}_{an})$ is injective it is not surjective.

It is a stunning result that while we cannot answer these questions above in the general case, they have very definite and positive answers when X is a projective variety:

Theorem. (GAGA) *Let X be a projective variety. For any coherent analytic sheaf $\mathcal{F}$ over X^{an} there exists a unique coherent algebraic sheaf $\mathcal{F}_{alg}$ over X such that $(\mathcal{F}_{alg})^{an} = \mathcal{F}$. Furthermore, for any $k \geq 0$ there is a natural isomorphism of cohomology groups*

$$H^k(X, \mathcal{F}_{alg}) = H^k(X^{an}, \mathcal{F}).$$

As the above examples show, several of GAGA's statements become false when the projectivity condition on X is relaxed. Fans of counterexamples can find several more in [Har77], but we do not have the necessary terminology to state most of them here. In short, none of GAGA's statements is correct for a general algebraic variety.

Before we can begin the proof we need to remind ourselves of a few facts on the sheaves of the tautological line bundles on $\mathbb{P}^n$, and we must establish versions of Cartan's theorems A and B for coherent analytic sheaves on compact complex spaces. These results will be instrumental to the proof of GAGA.

4.1 The sheaves $\mathcal{O}(d)$

Our treatment of the sheaves $\mathcal{O}(d)$ is rather informal. Proofs of nontrivial statements in this section may be found in [Dem07], [AG74], [Ser55] and [Ser56].

Definition 4.1.1. The *tautological line bundle* on $\mathbb{P}^n$ is the subbundle of the trivial vector bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$ defined by

$$\mathcal{O}(-1)_{[z]} = \mathbb{C} \cdot z \subset \mathbb{C}^{n+1}$$

If $U_j = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}$ is the usual open covering of $\mathbb{P}^n$, then the transition functions of $\mathcal{O}(-1)$ on $U_j \cap U_k$ are given by $g_{jk}([z]) = z_k/z_j$. The sets U_j are open in both the Zariski and usual topologies and the transition functions g_{jk} are both regular and holomorphic, so $O(-1)$ is both defined as an algebraic and holomorphic line bundle. In the case of ambiguity we write $\mathcal{O}_{alg}(-1)$ or $\mathcal{O}_{an}(-1)$ to specify which structure $\mathcal{O}(-1)$ has.

Definition 4.1.2. For any $d \in \mathbb{Z}$ we define $\mathcal{O}(1) = \mathcal{O}(-1)^*$, $\mathcal{O}(-d) = \mathcal{O}(-1)^{\otimes d}$ and $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$ for $d > 0$, and we set $\mathcal{O}(0)$ equal to the trivial line bundle $\mathbb{C}$.

Note that all $\mathcal{O}(d)$ can be considered as either algebraic or holomorphic line bundles. It is clear from the definition that the line bundles $\mathcal{O}(d)$ satisfy

$$\mathcal{O}(d+e) = \mathcal{O}(d) \otimes \mathcal{O}(e)$$

for all $d, e \in \mathbb{Z}$.

As per usual, we make no distinction between a line bundle and its sheaf of sections. In an abuse of notation we denote the sheaf of sections of $\mathcal{O}(d)$ again by $\mathcal{O}(d)$; when we want to precise whether this is the sheaf of regular or holomorphic sections we write $\mathcal{O}_{alg}(d)$ or $\mathcal{O}_{an}(d)$ as before. We note that $\mathcal{O}(0)$ is the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$ and that $(\mathcal{O}_{alg}(d))^{an} = \mathcal{O}_{an}(d)$ for any $d \in \mathbb{Z}$.

The sheaf $\mathcal{O}(d)$ is the sheaf of sections of a line bundle and is thus invertible, i.e. locally isomorphic to $\mathcal{O}$. It follows that $\mathcal{O}(d)$ is coherent and flat, as tensoring an exact sequence of sheaves of $\mathcal{O}$-modules is locally the same as tensoring it by $\mathcal{O}$, which changes nothing. Note also that as the sheaves $\mathcal{O}(d)$ are invertible, then they are locally isomorphic to each other.

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the usual projection, as either a regular or holomorphic function. It is useful to know that for any open $U \subset \mathbb{P}^n$ we have

$$\mathcal{O}(d)(U) = \{f \in \mathcal{O}(\pi^{-1}(U)) \mid f(\lambda z) = \lambda^d f(z) \text{ for all } \lambda \in \mathbb{C}^*\}$$

We can prove this equality by noting that the sets on the right side of the equation define an invertible sheaf over $\mathbb{P}^n$ which has the same transition functions as $\mathcal{O}(d)$.

The canonical sheaf morphism $\mathcal{O}_{alg}(d) \to \mathcal{O}_{an}(d)$ induces a morphism of the sheaf cohomology groups. The first part of the following proposition is proved in Serre's GAGA paper by chasing cohomology sequences, but it is also proved in [AG74] by direct calculations which give the second part as well:

Proposition 4.1.3. *For any $k \geq 0$ and $d \in \mathbb{Z}$ there is a canonical isomorphism $H^k(\mathbb{P}^n, \mathcal{O}_{alg}(d)) \simeq H^k(\mathbb{P}^n, \mathcal{O}_{an}(d))$. For $d \geq -n$, both groups are zero.*

We finish this section by establishing a certain short exact sequence for any coherent sheaf $\mathcal{F}$ over $\mathbb{P}^n$, which will be quite important for the proof of GAGA.

Remark — Let t be a non-zero linear form on $\mathbb{C}^{n+1}$, i.e. a function of the form $t(z) = a_0 z_0 + \ldots + a_n z_n$, where $a_j \in \mathbb{C}$ and not all a_j are zero. As $t(\lambda z) = \lambda t(z)$ for any $\lambda \in \mathbb{C}$ the set $A = \{[z] \in \mathbb{P}^n \,|\, t([z]) = 0\}$ is well defined and analytic, and we can form its ideal sheaf $\mathcal{I}_A$. It follows from the definition of $\mathcal{O}(-1)$ that the morphism $\mathcal{O}(-1) \to \mathcal{I}_A$ defined by multiplication by t is a sheaf isomorphism. In particular, $\mathcal{I}_A$ is invertible, and we have an exact sequence

$$0 \to \mathcal{I}_A \to \mathcal{O} \to \mathcal{O}_A \to 0 \tag{4.1}$$

Now let $\mathcal{F}$ be a coherent sheaf over $\mathbb{P}^n$. If $\mathcal{F}_A$ is the restriction of $\mathcal{F}$ to A, then it follows from the definition of analytic inverse image of a sheaf (i.e. the modified inverse image operation from definition 1.3.4 in the case of analytic sheaves) that $\mathcal{F}_A = \mathcal{F} \otimes \mathcal{O}_A$. Tensoring through (4.1) by $\mathcal{F}$ we get a short exact sequence

$$0 \to \mathcal{I}_A \cdot \mathcal{F} \to \mathcal{F} \to \mathcal{F}_A \to 0$$

where $\mathcal{I}_A \cdot \mathcal{F}$ is the image of $\mathcal{I}_A \otimes \mathcal{F}$ in $\mathcal{F}$. Recall that $\mathcal{I}_A$ is invertible, so locally we have $\mathcal{I}_A \otimes \mathcal{F} = \mathcal{F}$. It follows that $\mathcal{I}_A \otimes \mathcal{F} \to \mathcal{F}$ is actually injective, so we have a short exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{O}(-1) \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_A \to 0$$

Now set $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(d)$ for any $d \in \mathbb{Z}$. The sheaves $\mathcal{F}(d)$ are coherent, we have that $\mathcal{F}_A(d) = \mathcal{F}(d) \otimes \mathcal{O}_A$, and as the sheaves $\mathcal{O}(d)$ are flat the sequence

$$0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to \mathcal{F}_A(d) \to 0 \tag{4.2}$$

is exact. This last short exact sequence will be used in the proof of theorem A in the next section.

4.2 Theorems A and B

Let $\mathcal{F}$ be a coherent analytic sheaf over $\mathbb{P}^n$. For any $d \in \mathbb{Z}$ we set $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(d)$. As both $\mathcal{F}$ and $\mathcal{O}(d)$ are coherent, the sheaf $\mathcal{F}(d)$ is coherent. The key to the proof of GAGA is the following two theorems:

Theorem A. *Let $\mathcal{F}$ be a coherent analytic sheaf over $\mathbb{P}^n$. There exists a $d_0 \in \mathbb{Z}$ such that for every $d \geq d_0$ the sheaf $\mathcal{F}(d)$ is generated by its global sections, i.e. for every $x \in \mathbb{P}^n$ there exist sections $f_1, \ldots, f_p \in H^0(\mathbb{P}^n, \mathcal{F}(d))$ which generate $\mathcal{F}(d)_x$ as a $\mathcal{O}$-module.*

Theorem B. *Let $\mathcal{F}$ be a coherent analytic sheaf over $\mathbb{P}^n$. There exists a $d_0 \in \mathbb{Z}$ such that $H^k(\mathbb{P}^n, \mathcal{F}(d)) = 0$ for every $k \geq 1$ and every $d \geq d_0$.*

These theorems are versions of Cartan's theorems A and B, which give the same results for a coherent sheaf $\mathcal{F}$ on a strongly pseudoconvex manifold, but without any reference to the sheaves $\mathcal{F}(d)$.

The proofs of these theorems will proceed in several steps. We will show that theorem B follows from theorem A, establish a lemma for use in the proof of theorem A, and finally prove theorem A. Let us begin by proving a proposition that will be used in the proofs of theorem B and GAGA:

Proposition 4.2.1. *Assume theorem A holds and let $\mathcal{F}$ be a coherent analytic sheaf over $\mathbb{P}^n$. Then there exists an exact sequence*

$$\mathcal{O}(e)^q \to \mathcal{O}(d)^p \to \mathcal{F} \to 0$$

for some $d, e \in \mathbb{Z}$ and $p, q \geq 0$.

Proof: According to theorem A there is an exact sequence $\mathcal{O}^p \to \mathcal{F}(-d) \to 0$, and if we tensor through the sequence with $\mathcal{O}(d)$ we obtain the exact sequence $\mathcal{O}(d)^p \to \mathcal{F} \to 0$. If we do the same to the kernel of the map in this sequence, we get $\mathcal{O}(e)^q \to \mathcal{O}(d)^p \to \mathcal{F} \to 0$ as we wanted. □

We can now show that theorem B is a consequence of theorem A:

Proof of theorem B: By proposition 4.2.1 there exists an exact sequence

$$\mathcal{O}(d')^q \to \mathcal{O}(d)^p \xrightarrow{\varphi} \mathcal{F} \to 0$$

from which we get the short exact sequence

$$0 \to \operatorname{Ker} \varphi \to \mathcal{O}(d)^p \to \mathcal{F} \to 0 \tag{4.3}$$

and thus

$$0 \to \operatorname{Ker} \varphi(e) \to \mathcal{O}(d+e)^p \to \mathcal{F}(e) \to 0 \tag{4.4}$$

exact for any $e \in \mathbb{Z}$. The proof will now proceed by descending induction on the order of the cohomology group $H^k(\mathbb{P}^n, \mathcal{F}(e))$. We can start the induction because $H^k(\mathbb{P}^n, \mathcal{F}(e)) = 0$ for $k > \dim_{\mathbb{R}} \mathbb{P}^n = 2n$.

Let k be given and suppose the result holds for $k+1$. By proposition 4.1.3 we know that $H^k(\mathbb{P}^n, \mathcal{O}(d+e)) = 0$ for any $d + e \geq -n$, so by the induction step we can find a d_0 such that

$$H^k(\mathbb{P}^n, \mathcal{O}(d+e)) = H^{k+1}(\mathbb{P}^n, \operatorname{Ker} \varphi(e)) = 0$$

for any $e \geq d_0$. From the exact sequence (4.4) we get an exact sequence

$$H^k(\mathbb{P}^n, \mathcal{O}(d+e)^p) \to H^k(\mathbb{P}^n, \mathcal{F}(e)) \to H^{k+1}(\mathbb{P}^n, \operatorname{Ker}\varphi(e))$$

for any e, and thus $H^k(\mathbb{P}^n, \mathcal{F}(e)) = 0$ for any $e \geq d_0$. □

For the proof of theorem A we need the following lemma, which in fact does most of the job for us:

Lemma 4.2.2. *Let $\mathcal{F}$ be a coherent analytic sheaf over $\mathbb{P}^n$. For each x in $\mathbb{P}^n$ there exists a d_0 such that $\mathcal{F}(d_0)_x$ is generated by its global sections.*

Proof: We proceed by induction on the dimension of $\mathbb{P}^n$. For $n = 0$ the space $\mathbb{P}^n$ is a point and any locally finitely generated sheaf over $\mathbb{P}^n$ is thus trivially generated by its global sections.

Let $n > 0$ be given and suppose the lemma holds on every $\mathbb{P}^k$ for $k < n$. Let $A \simeq \mathbb{P}^{n-1}$ be a hyperplane of $\mathbb{P}^n$ that passes through x. By remark 13 there is for any $d \in \mathbb{Z}$ a short exact sequence

$$0 \to \mathcal{F}(d-1) \to \mathcal{F}(d) \to \mathcal{F}_A(d) \to 0$$

which in turn gives rise to a long exact sequence

$$\begin{aligned} 0 \to H^0(\mathbb{P}^n, \mathcal{F}(d-1)) \to \quad & H^0(\mathbb{P}^n, \mathcal{F}(d)) \quad \to H^0(\mathbb{P}^{n-1}, \mathcal{F}_A(d)) \to \\ \to H^1(\mathbb{P}^n, \mathcal{F}(d-1)) \to \quad & H^1(\mathbb{P}^n, \mathcal{F}(d)) \quad \to H^1(\mathbb{P}^{n-1}, \mathcal{F}_A(d)) \to \dots \end{aligned}$$

By the induction hypothesis theorem A is true for sheaves over $\mathbb{P}^{n-1}$, so theorem B is also true for sheaves over $\mathbb{P}^{n-1}$. This lets us conclude that there exists a d_1 such that $H^1(\mathbb{P}^{n-1}, \mathcal{F}_A(d)) = 0$, i.e. that $H^1(\mathbb{P}^n, \mathcal{F}(d-1)) \to H^1(\mathbb{P}^n, \mathcal{F}(d))$ is surjective, for all $d \geq d_1$.

We now have a long sequence with surjective maps

$$H^1(\mathbb{P}^n, \mathcal{F}(d_1 - 1)) \to H^1(\mathbb{P}^n, \mathcal{F}(d_1)) \to \dots H^1(\mathbb{P}^n, \mathcal{F}(d)) \to \dots$$

Let us recall the Cartan-Serre theorem of chapter 3, which says that these vector spaces are all finite dimensional. As each map in the sequence is surjective, we conclude that the sequence of integers $(\dim H^1(\mathbb{P}^n, \mathcal{F}(d)))_{d \geq d_1 - 1}$ is decreasing. The sequence is also bounded below by 0, so there exists a d_2 such that it is stationary for all $d \geq d_2$, which means that the maps $H^1(\mathbb{P}^n, \mathcal{F}(d)) \to H^1(\mathbb{P}^n, \mathcal{F}(d+1))$ are isomorphisms for all $d \geq d_2$.

Our long exact sequence above then reduces to a short exact sequence

$$0 \to H^0(\mathbb{P}^n, \mathcal{F}(d-1)) \to H^0(\mathbb{P}^n, \mathcal{F}(d)) \to H^0(\mathbb{P}^{n-1}, \mathcal{F}_A(d)) \to 0$$

for every $d \geq d_2$. By the induction hypothesis there exists a $d_0 \geq d_2$ such that $\mathcal{F}_A(d_0)_x$ is generated by $H^0(\mathbb{P}^{n-1}, \mathcal{F}_A(d_0))$ over $\mathcal{O}_A$. We claim that $\mathcal{F}(d_0)_x$ is then also generated by its global sections:

To simplify the notation, we put $A = \mathcal{O}_{\mathbb{P}^n,x}$, $E = \mathcal{F}(d_0)_x$, $\mathfrak{p} = \mathcal{I}_{A,x}$ and let F be the sub-A-module of E generated by $H^0(\mathbb{P}^n, \mathcal{F}(d_0))$. Our goal is to show that $E = F$. We have

$$\mathcal{F}_A(d_0)_x = \mathcal{F}(d_0)_x \otimes_A \mathcal{O}_{A,x} = E \otimes_A A/\mathfrak{p} = E/\mathfrak{p}E,$$

where the first equality is by remark 13, the second is valid because $\mathcal{O}_{A,x} = \mathcal{O}_{\mathbb{P}^n,x}/\mathcal{I}_{A,x}$ and the third follows from general properties of tensor products. Furthermore, by the surjection of $H^0(\mathbb{P}^n, \mathcal{F}(d_0)) \to H^0(\mathbb{P}^{n-1}, \mathcal{F}_A(d_0))$ we know that the image of F in $E/\mathfrak{p}E$ generates $E/\mathfrak{p}E$. This means that $E = F + \mathfrak{p}E$, from which we get $E = F + \mathfrak{m}E$, where $\mathfrak{m}$ is the unique maximal ideal of E. By corollary A.1.2 to Nakayama's lemma, we get that $E = F$, i.e. the global sections of $\mathcal{F}(d_0)$ generate $\mathcal{F}(d_0)_x$. □

Finally we prove theorem A:

Proof of theorem A: For every x in $\mathbb{P}^n$ there is a d_x such that $H^0(\mathbb{P}^n, \mathcal{F}(d_x))$ generates $\mathcal{F}(d_x)_x$. The sheaf $\mathcal{F}(d_x)$ is locally finitely generated, so there are global sections $f_{x,1}, \ldots, f_{x,N}$ which generate the stalk $\mathcal{F}(d_x)_y$ in a neighborhood V_x of x.

Take a neighborhood $W_x = \{[z] \in \mathbb{P}^n \mid z_j \neq 0\}$ of x. Then the sheaf $\mathcal{O}(1)_y$ is generated its global sections on W_x, we can even take the global sections to be the ones corresponding to the coordinate functions $z_0, \ldots, z_n$ on $\mathbb{C}^{n+1} \setminus \{0\}$. Then both $\mathcal{F}(d_x)_y$ and $\mathcal{O}(1)_y$ are generated by their global sections on $U_x := V_x \cap W_x$.

Now note that if $\mathcal{G}$ and $\mathcal{H}$ are generated by their global sections on U, then $\mathcal{G} \otimes \mathcal{H}$ is also generated by its global sections on U; if $(g_j)_{j \in J}$ generate $\mathcal{G}$ and $(h_k)_{k \in K}$ generate $\mathcal{H}$ on U, then $(g_j \otimes h_k)_{j \in J, k \in K}$ generate $\mathcal{G} \otimes \mathcal{H}$ on U. Thus $\mathcal{F}(d)_y$ is generated on U_x by its global sections for any $d \geq d_x$.

Finally, cover $\mathbb{P}^n$ by open U_x such that the global sections of $\mathcal{F}(d)$ generate its stalks on U_x for any $d \geq d_x$, and take a finite subcovering $U_{x_1}, \ldots, U_{x_m}$ of $\mathbb{P}^n$. Then the global sections of $\mathcal{F}(d)$ generate its stalks on all of $\mathbb{P}^n$ for all $d \geq d_0 := \max\{d_{x_1}, \ldots, d_{x_m}\}$. □

4.3 The proof of GAGA

Theorem 4.3.1. (GAGA) *Let X be a projective variety. For any coherent analytic sheaf $\mathcal{F}$ over X there exists a unique coherent algebraic sheaf*

$\mathcal{F}_{alg}$ over X such that $(\mathcal{F}_{alg})^{an} = \mathcal{F}$. Furthermore, for any $k \geq 0$ there is a natural isomorphism of cohomology groups

$$H^k(X, \mathcal{F}_{alg}) = H^k(X, \mathcal{F}).$$

Remark — It is enough to prove GAGA for the case $X = \mathbb{P}^n$. For the result on the cohomology groups, this follows from proposition 1.3.1 and remark 2. The statement on uniqueness will follow from the other two, and for existence we reason as follows:

Extend the sheaf $\mathcal{F}$ by zero to all of $\mathbb{P}^n$ and denote the extension by $\mathcal{F}^{\mathbb{P}^n}$, the extension will be a coherent analytic sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules by propositions 1.3.2 and 1.3.3. Suppose there exists a coherent algebraic sheaf $\mathcal{G}$ on $\mathbb{P}^n$ such that $\mathcal{G}^{an} = \mathcal{F}^{\mathbb{P}^n}$. If $i : X \hookrightarrow \mathbb{P}^n$ is the injection, then $\mathcal{G}_{|X} := i^*\mathcal{G}$ will be a coherent algebraic sheaf over X such that $(\mathcal{G}_{|X})^{\mathbb{P}^n} = \mathcal{G}$. By proposition 2.3.6 we have that

$$(\mathcal{G}_{|X}^{an})^{\mathbb{P}^n} = (\mathcal{G}_{|X}^{\mathbb{P}^n})^{an} = \mathcal{G}^{an} = \mathcal{F}^{\mathbb{P}^n}$$

and thus $\mathcal{G}_{|X}^{an} = \mathcal{F}$ by restriction to X.

Proof of GAGA: The proof will proceed by three steps. First we construct the sheaf $\mathcal{F}_{alg}$, then we show the isomorphism of cohomology groups, and finally we prove uniqueness.

Step 1: Existence.

By proposition 4.2.1 there exists an exact sequence

$$\begin{array}{ccccccc} \mathcal{O}_{an}(e)^q & \stackrel{\varphi}{\to} & \mathcal{O}_{an}(d)^p & \to & \mathcal{F} & \to & 0 \\ \uparrow & & \uparrow & & & & \\ \mathcal{O}_{alg}(e)^q & & \mathcal{O}_{alg}(d)^p & & & & \end{array}$$

where the vertical arrows are the canonical morphisms. Note that $\mathcal{F} = \operatorname{Coker}\varphi$. We would like to define a morphism $\mathcal{O}_{alg}(e)^q \to \mathcal{O}_{alg}(d)^p$ which makes this diagram commutative, as then we could define $\mathcal{F}_{alg}$ as the cokernel of that morphism. We will show that the morphism φ is in fact the analytification of an algebraic $\varphi_{alg} : \mathcal{O}_{alg}(e)^q \to \mathcal{O}_{an}(d)^p$, which will certainly make our diagram commutative.

The morphism φ is an element of $H^0(\mathbb{P}^n, \operatorname{Hom}_{\mathcal{O}_{an}}(\mathcal{O}_{an}(e)^q, \mathcal{O}_{an}(d)^p))$. We calculate that

$$\operatorname{Hom}_{\mathcal{O}_{an}}(\mathcal{O}_{an}(e)^q, \mathcal{O}_{an}(d)^p) = \mathcal{O}_{an}(-e)^q \otimes \mathcal{O}_{an}(d)^p = \bigoplus_{i=1}^{pq} \mathcal{O}_{an}(d-e)$$

so φ is in fact an element of

$$\bigoplus_{i=1}^{pq} H^0(\mathbb{P}^n, \mathcal{O}_{an}(d-e)).$$

But by proposition 4.1.3 we have $H^0(\mathbb{P}^n, \mathcal{O}_{an}(d-e)) = H^0(\mathbb{P}^n, \mathcal{O}_{alg}(d-e))$, and the calculations above made again in reverse for $\mathcal{O}_{alg}$ we find that there is a unique element φ_{alg} of $H^0(\mathbb{P}^n, \mathrm{Hom}_{\mathcal{O}_{alg}}(\mathcal{O}_{alg}(e)^q, \mathcal{O}_{alg}(d)^p))$ such that $\varphi_{alg} \mapsto \varphi$ via the canonical morphism.

We thus have a coherent algebraic sheaf $\mathcal{F}_{alg} := \mathrm{Coker}\, \varphi_{alg}$ which fits into the exact sequence

$$\mathcal{O}_{alg}(e)^q \stackrel{\varphi_{alg}}{\to} \mathcal{O}_{an}(d)^p \to \mathcal{F}_{alg} \to 0$$

Analytifying we find that

$$\mathcal{O}_{an}(e)^q \stackrel{\varphi}{\to} \mathcal{O}_{an}(d)^p \to (\mathcal{F}_{alg})^{an} \to 0$$

is exact, so $(\mathcal{F}_{alg})^{an} = \mathrm{Coker}\, \varphi = \mathcal{F}$.

Step 2: Cohomology.

We will prove that the natural maps $H^k(\mathbb{P}^n, \mathcal{F}_{alg}) \to H^k(\mathbb{P}^n, \mathcal{F})$ induced by the canonical map $\mathcal{F}_{alg} \to \mathcal{F}$ are isomorphisms by descending induction on k. Note that we can start the induction as

$$H^k(\mathbb{P}^n, \mathcal{F}_{alg}) = H^k(\mathbb{P}^n, \mathcal{F}) = 0$$

for $k > \dim_{\mathbb{R}} \mathbb{P}^n = 2n$[1].

From the construction of $\mathcal{F}_{alg}$ we have a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{H}_{alg} & \to & \mathcal{O}_{alg}(d)^p & \to & \mathcal{F}_{alg} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{H} & \to & \mathcal{O}_{an}(d)^p & \to & \mathcal{F} & \to & 0 \end{array}$$

where $\mathcal{H}_{alg} := \mathrm{Im}\, \varphi_{alg}$, $\mathcal{H} := \mathrm{Im}\, \varphi$ and the horizontal arrows are exact and everything commutes. This gives rise to a commutative diagram with long

[1]As my advisor pointed out to me during my defense, this has nothing to do with the topological dimension of the space in the algebraic case. Indeed, the Zariski-topological dimension of $\mathbb{P}^n$ is infinite as the intersection of any finite number of open sets is non-empty. In the algebraic case (and the analytic as well, if we so wish) we obtain the result by calculating the comohology groups using alternating Čech cochains, see [Dem07].

exact horizontal sequences (we don't write $\mathbb{P}^n$ to fit everything in one line)

$$\begin{array}{ccccccccc} H^k(\mathcal{H}_{alg}) & \to & H^k(\mathcal{O}_{alg}(d)^p) & \to & H^k(\mathcal{F}_{alg}) & \to & H^{k+1}(\mathcal{H}_{alg}) & \to & H^{k+1}(\mathcal{O}_{alg}(d)^p) \\ (1)\downarrow & & (2)\downarrow & & (3)\downarrow & & (4)\downarrow & & (5)\downarrow \\ H^k(\mathcal{H}) & \to & H^k(\mathcal{O}_{an}(d)^p) & \to & H^k(\mathcal{F}) & \to & H^{k+1}(\mathcal{H}) & \to & H^{k+1}(\mathcal{O}_{an}(d)^p) \end{array}$$

By the induction hypothesis arrows (4) and (5) are isomorphisms, and we already know that (2) is an isomorphism, so (3) is surjective by the five-lemma.

Now, set $\mathcal{G}_{alg} := \operatorname{Ker} \varphi_{alg}$, $\mathcal{G} := \operatorname{Ker} \varphi$ and look at the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{G}_{alg} & \to & \mathcal{O}_{alg}(e)^q & \to & \mathcal{H}_{alg} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{G} & \to & \mathcal{O}_{an}(e)^q & \to & \mathcal{H} & \to & 0 \end{array}$$

This gives another long exact commutative diagram

$$\begin{array}{ccccccccc} H^k(\mathcal{G}_{alg}) & \to & H^k(\mathcal{O}_{alg}(e)^q) & \to & H^k(\mathcal{H}_{alg}) & \to & H^{k+1}(\mathcal{G}_{alg}) & \to & H^{k+1}(\mathcal{O}_{alg}(e)^q) \\ (1')\downarrow & & (2')\downarrow & & (3')\downarrow & & (4')\downarrow & & (5')\downarrow \\ H^k(\mathcal{G}) & \to & H^k(\mathcal{O}_{an}(e)^q) & \to & H^k(\mathcal{H}) & \to & H^{k+1}(\mathcal{G}) & \to & H^{k+1}(\mathcal{O}_{an}(e)^q) \end{array}$$

where the arrows (4') and (5') are isomorphisms by the induction hypothesis, and we again know that (2') is an isomorphism. By another application of the five-lemma we find that (3') = (1) is surjective.

But then arrows (2) and (4) are injective and arrow (1) is surjective, so (3) is injective by the five-lemma, and is therefore an isomorphism, which is what we wanted to prove.

Step 3: Uniqueness.

Suppose that $\mathcal{G}_{alg}$ is another coherent algebraic sheaf such that there exists an isomorphism $\varphi : (\mathcal{F}_{alg})^{an} \to (\mathcal{G}_{alg})^{an}$. The morphism φ is a global section of the coherent analytic sheaf

$$\operatorname{Hom}_{Oan}((\mathcal{F}_{alg})^{an}, (\mathcal{G}_{alg})^{an}) \simeq (\operatorname{Hom}_{Oalg}(\mathcal{F}_{alg}, \mathcal{G}_{alg}))^{an}, \tag{4.5}$$

where the sheaves are isomorphic by corollary 2.3.5, so by step 2 there exists an algebraic morphism $\varphi_{alg} : \mathcal{F}_{alg} \to \mathcal{G}_{alg}$ such that $(\varphi_{alg})^{an} = \varphi$. By the same reasoning there exists an algebraic morphism $\psi_{alg} : \mathcal{G}_{alg} \to \mathcal{F}_{alg}$ such that $(\psi_{alg})^{an} = \varphi^{-1}$. Then we find that

$$(\varphi_{alg} \circ \psi_{alg})^{an} = (\varphi_{alg})^{an} \circ (\psi_{alg})^{an} = \mathrm{id}_{(\mathcal{F}_{alg})^{an}}$$

and we conclude that $\varphi_{alg} \circ \psi_{alg} = \mathrm{id}_{\mathcal{F}_{alg}}$ because of the isomorphism (4.5). In the same way we see that $\psi_{alg} \circ \varphi_{alg} = \mathrm{id}_{\mathcal{G}_{alg}}$, so $\psi_{alg} = \varphi_{alg}^{-1}$ and

$$\varphi_{alg} : \mathcal{F}_{alg} \to \mathcal{G}_{alg}$$

is an isomorphism. The proof of GAGA is complete. □

4.4 On applying GAGA

We now have a very strong theorem in our hands, and it is natural to ask ourselves when we can apply it. In effect, we want many examples of coherent sheaves, and conditions on complex spaces which guarantee that they arise as analytifications of projective varieties.

First off, we have several examples of coherent sheaves: The structure sheaf of either a complex space or analytic variety is coherent. All vector bundles, or rather their sheaves of sections, are coherent. The ideal sheaves of subspaces and subvarieties are coherent. And due to the results of chapter 1 the class of coherent sheaves is closed under the application of a great deal of algebraic operations.

Now, we know that if X and Y are algebraic varieties or complex spaces, $f : X \to Y$ is a morphism and $\mathcal{G}$ is a coherent sheaf over Y, then the analytic or algebraic inverse image $f^*\mathcal{G}$ is a coherent sheaf over X. It is not always the case that if $\mathcal{F}$ is a coherent sheaf over X, then the direct image $f_*\mathcal{F}$ is coherent over Y. A necessary condition for the direct image to be coherent was given by Grothendieck in the algebraic case and by Grauert and Remmert in the analytic situation: If f is proper then $f_*\mathcal{F}$ is coherent. See [GR84] for a proper statement and proof of the analytic case.

As we said it is also natural to ask what conditions a complex space has to satisfy to be a projective variety. This is a hard question for general complex spaces, but we have satisfactory answers in the cases of analytic sets and complex manifolds. Chow proved the following theorem in 1949, which we can obtain as a quick corollary of GAGA:

Theorem 4.4.1. (Chow) *Let A be an analytic subset of a projective variety X. Then A is algebraic.*

Proof: We want to show that A is closed in the Zariski topology. Recall that A is equal to the support of $\mathcal{O}_X/\mathcal{I}_A$, which is a coherent analytic sheaf over X^{an}. By GAGA there exists a coherent algebraic sheaf $\mathcal{F}$ over X such that $\mathcal{F}^{an} = \mathcal{O}_X/\mathcal{I}_A$. But

$$\operatorname{Supp} \mathcal{F} = \operatorname{Supp} \mathcal{O}_X/\mathcal{I}_A = A,$$

and Supp $\mathcal{F}$ is Zariski-closed because $\mathcal{F}$ is locally finitely generated. □

Combined with an embedding theorem of Kodaira from 1954, Chow's result gives the following strong theorem, proven for example in [Dem07]:

Theorem 4.4.2. (Kodaira) *Let X be a compact complex manifold. The following conditions are equivalent:*

a) X is a projective variety, i.e. X can be embedded as an algebraic variety in $\mathbb{P}^N$ for some N.

b) X carries a positive line bundle L.

c) X carries a Hodge metric, i.e. a Kähler metric ω with rational cohomology class $\{\omega\} \in H^2(X, \mathbb{Q})$.

There is a useful corollary of Kodaira's theorem. Its proof is not hard, but it involves techniques from Hodge theory which we have not established, so we refer again to Demailly's book for the proof:

Corollary 4.4.3. *Let (X, ω) be a compact Kähler manifold. If $H^2(X, \mathcal{O}) = 0$ then X is projective.*

Appendix A

Some algebraic results

A.1 Nakayama's lemma

The following facts were used in the proof of lemma 4.2.2:

Recall that a ring R is local if it has a unique maximal ideal $\mathfrak{m}$. Let R be a local noetherian commutative ring.

Lemma A.1.1. (Nakayama) *Let E be a finitely generated R-module. If $\mathfrak{m}E = E$, then $E = 0$.*

Proof: By induction on the number of generators of E. Suppose E is generated by n elements $x_1, \ldots, x_n$. Then $\mathfrak{m}E = E$ means that there exist $a_1, \ldots a_n$ in $\mathfrak{m}$ such that

$$x_n = a_1 x_1 + \ldots + a_n x_n$$

and thus $(1 - a_n)x_n = a_1 x_1 + \ldots + a_{n-1} x_{n-1}$. As $1 - a_n$ is a unit in R, then E is generated by $(n-1)$ elements. □

Corollary A.1.2. *Let E be a finitely generated R-module and $F \subset E$ a submodule. If $E = F + \mathfrak{m}E$, then $E = F$.*

Proof: Note that $E/F = \mathfrak{m}E/F = \mathfrak{m}(E/F)$ and apply Nakayama's lemma. □

Remark — Let $\mathcal{O}_n$ be the sheaf of germs of holomorphic functions of n variables at 0. It is a consequence of the Weierstrass division theorem that $\mathcal{O}_n$ is noetherian, and it is local with maximal ideal $\mathfrak{m} = \{f \in \mathcal{O}_n \mid f(0) = 0\}$. The proofs of these facts may be found in [Dem07] or [GR65].

A.2 The Krull topology of $\mathcal{O}_n$-modules

Some results used in chapter 3 may be found in this small extract of [Dem07]. We only state the results we used, and refer to the source for their proofs.

Let R be a noetherian local commutative ring with maximal ideal $\mathfrak{m}$.

Lemma A.2.1. (Krull) *Let F be a finitely generated R-module and let E be a submodule. Then*

a) $\bigcap\limits_{k\geq 0} \mathfrak{m}^k F = \{0\}$.

b) $\bigcap\limits_{k\geq 0} (E + \mathfrak{m}^k F) = E$.

Now assume that $R = \mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$, and that $\mathfrak{m} = (z_1, \dots, z_n)$. Then $\mathcal{O}_n/\mathfrak{m}^k$ is a finite dimensional $\mathbb{C}$-vector space generated by the monomials z^α for $|\alpha| < k$. It follows that $E/\mathfrak{m}^k E$ is a finite dimensional vector space for any finitely generated $\mathcal{O}_n$-modules E. As $\cap_{k\geq 0}\mathfrak{m}^k E = \{0\}$ by the lemma, there is an injection

$$E \hookrightarrow \prod_{k\geq 0} E/\mathfrak{m}^k E$$

We give E the Hausdorff topology induced by the product, that is the weakest topology that makes all the projections $E \to E/\mathfrak{m}^k E$ continuous. This topology is called the *Krull topology* on E. For $E = \mathcal{O}_n$ this is the topology of simple convergence on coefficients, defined by the collection of semi-norms $\sum c_\alpha z^\alpha \mapsto |c_\alpha|$. The theorem we used in chapter 3 was:

Theorem A.2.2. *Let $E \subset F$ be finitely generated $\mathcal{O}_n$-modules. Then E is closed in F.*

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